

# A utility maximization problem with state constraint and non-concave technology

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## Abstract

We consider an optimal control problem arising in the context of economic theory of growth, on the lines of the works by Skiba (1978) and Askenazy - Le Van (1999).

The economic framework of the model is intertemporal infinite horizon utility maximization. The dynamics involves a state variable representing total endowment of the social planner or average capital of the representative dynasty. From the mathematical viewpoint, the main features of the model are the following: (i) the dynamics is an increasing, unbounded and not globally concave function of the state; (ii) the state variable is subject to a static constraint; (iii) the admissible controls are merely locally integrable in the right half-line. Such assumptions seem to be weaker than those appearing in most of the existing literature.

We give a direct proof of the existence of an optimal control for any initial capital  $k_0 \geq 0$  and we carry on a qualitative study of the value function; moreover, using dynamic programming methods, we show that the value function is a continuous viscosity solution of the associated Hamilton-Jacobi-Bellman equation.

**Keywords:** Optimal control, utility maximization, convex-concave production function, Hamilton Jacobi Bellman equation, viscosity solutions.

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## 1 Introduction

Utility maximization problems represent a fundamental part of modern economic growth models, since the works by Ramsey (1928), Romer (1986), Lucas (1988), Barro and Sala-i-Martin (1999). These models aim to formalize the dynamics of an economy throughout the quantitative description of the consumers' behaviour. Consumers are seen as homogeneous entities, as far as their operative decisions are concerned; hence the time series of their consuming choices, or consumption path, is represented by a single function, and they as a collective are named after *social planner*, or simply *agent*.

The agent's purpose is to maximize the utility in function of the series of the consumption choices in a fixed time interval; this can be finite or more often (as far as economic growth literature is concerned) infinite.

From the application viewpoint, the target of the analysis is the study of the optimal – in relation to this utility functional – trajectories: regularity, monotonicity, asymptotic behaviour properties and similar are expected to be investigated. Hence good existence results are specially needed, as well as handy sufficient and necessary conditions for the optimum.

These problems are treated mathematically as optimal control problems; often external reasons such as the pursuit of more empirical description power imply the presence of additional state constraints, which we may call “static” since they do not involve the derivative of the state variable.

It is worth noticing that the introduction of the static state constraints usually makes the problem quite harder (and it is sometimes considered extraneous to the usual setting of control theory). As an example, we see that the main properties of optimal trajectories are still not characterized in recent literature, at least in the case of non-concave production function.

Hence this kind of program is quite complex, especially in the latter case – and has to be dealt with in many phases. Here we undertake the work providing an existence result and various necessary conditions related to the Hamilton-Jacobi-Bellman problem (HJB), remembering Skiba (1978) and Askenazy - Le Van (1999) and developing part of the studies carried on by F. Gozzi and D. Fiaschi (2009).

Some technical difficulties arise as an effect of the generality of the hypothesis on the data, which are supposed to be the reason of the versatility and wide-range applicability of this model.

First, the dynamics contains a convex-concave function representing production. It is well known that the presence of non-concavity in an optimization problem can lead to many difficulties in establishing the necessary and sufficient conditions for the optimum, as well as in examining the regularity properties of the value function.

Secondly, the above mentioned presence of the static state constraint makes any admissibility proof much more complicated than usual.

As a third relevant feature, we require that the admissible controls are not more than locally integrable in the positive half-line: this is the maximal class if one wants the control strategy to be a function and the state equation to have solution. This is a weak regularity requirement which is of very little help; conversely it generates unexpected issues in various respects.

We can summarize the main criticalities entailed by these three traits as follows:

1. Certain questions appear that in other "bounded-control" models are not even present, such as the finiteness of the value function and the well-posedness of the Hamiltonian problem, i.e.

the question whether the value function is a viscosity solution to the HJB equation. The notion of viscosity solution can be characterized both in terms of super- and sub-differentials and of test functions; in any case these auxiliary tools must match the necessary restrictions to the domain of the Hamiltonian function, at least for the solutions we are interested in verifying. Fortunately, we are able to prove certain regularity properties of the value function ensuring that this is the case.

2. The problem of the existence of an optimal control strategy (for every fixed initial state) lacks of couplings in the traditional literature such as Cesari, Zabczyk, Yong-Zhou. It is a natural idea to make use of the traditional compactness results, in order to generate a convergent approximation procedure. As we commit ourselves to deal with merely (locally) integrable control functions, the application of such compactness results is not straightforward. Indeed, a very careful preliminary work is needed, providing a uniform localization lemma. The procedure has then to be further refined so that we can find a limit function which is admissible in the sense that it satisfies the static state constraint, and whose functional is (not less then) the limit of the approximating sequence functionals.

3. Additional work to the usual proof of the fact that the value function indeed solves HJB is needed; in fact we use the uniform localization lemma which appears in the optimality construction: the fundamental Lemma 9.

4. The regularity property stated in Theorem 25.ii), which is necessary in order that the HJB problem is well-posed, not only requires optimal controls. It can be proven by a standard argument under the hypothesis that the admissible controls are locally bounded; in our case it shows again to be useful to come back to the preliminary tools (Lemmas 9 and 10) in order to move around the obstacle and have the result proven with merely integrable control functions.

The contents are consequently arranged: first, the reader will come across an introductory paragraph which intends to clear up the genesis of the model and the economic motivations for the assumptions.

Then comes a section dedicated to the preliminary results that are crucial for the development of the theory.

Afterwards, some basic properties of the value function are proven, such as its behaviour near the origin and near  $+\infty$ . These results require careful manipulations of the data and some standard results about ordinary differential equations, but do not require the existence of optimal control functions.

Next comes the section in which we prove the existence of an optimal control strategy for every initial state. Here we make wide use of the preliminary lemmas in association with a special diagonal procedure generating a weakly convergent sequence of controls from a family of sequences which, unlike in Ascoli-Arzelà's theorem, are not extracted neatly one from the other.

After providing the existence theorem, we are able to prove other important regularity properties of the value function (such as the Lipschitz-continuity in the closed intervals of  $(0, +\infty)$ ), using optimal controls.

Eventually we give an application of the methods of Dynamic Programming to our model. As mentioned before, the proof of the admissibility of the value function as a viscosity solution of HJB is made more complicated by the use of the preliminary lemmas, but it allows to obtain the result independently of the regularity of the Hamiltonian function, which contributes to make this problem peculiar and hopefully a source of further motives of scientific interest.

We note that the admissible controls, modelling the agent's consumptions, are supposed to be locally integrable also for representativeness purposes. Since we are able to reach a local boundedness result (the above mentioned Lemma 9), we could also have developed most of the optimum existence proof in  $L^2$ , and then come back to  $L^1$ . Since this space is our natural environment we have chosen to use the fact that, for a finite-measure space  $E$ , a sequence which is uniformly bounded in  $L^\infty(E)$  admits a subsequence which is weakly convergent in  $L^1(E)$  (this is an easy consequence of the Dunford-Pettis theorem).

## 2 The model

### 2.1 Qualitative description

We assume the existence of a representative dynasty in which all members share the same endowments and consume the same amount of a certain good. Our goal is to describe the dynamics of the capital accumulated by each member of the dynasty in an infinite-horizon period and to maximize its intertemporal utility (considered as a function of the quantity of good  $c$  that has been consumed). Clearly, consuming is seen as the agent's control strategy, and the set of consumption functions (over time) will be a superset of the set of the admissible control strategies.

First, we need a notion of instantaneous utility, depending on the consumptions, in order to define the inter-temporal utility functional. We will assume that instantaneous utility, which we denote by  $u$ , is a strictly increasing and strictly concave function of the consumptions, and that it is twice continuously differentiable. Moreover, we will assume the usual Inada's conditions, that is to say:

$$\lim_{c \rightarrow 0^+} u'(c) = +\infty, \quad \lim_{c \rightarrow +\infty} u'(c) = 0.$$

We will also use the following assumptions on  $u$ :

$$u(0) = 0, \quad \lim_{c \rightarrow +\infty} u(c) = +\infty.$$

With this material, we can define the inter-temporal utility functional, which, as usual, must include a (exponential) discount factor expressing time preference for consumption:

$$U(c(\cdot)) := \int_0^{+\infty} e^{-\hat{\rho}t} e^{nt} u(c(t)) dt \quad (1)$$

where  $\hat{\rho} \in \mathbb{R}$  is the rate of time preference and  $n \in \mathbb{R}$  is the growth rate of population. The number of members of the dynasty at time zero is normalized to 1.

### 2.2 Production function and constraints

We consider the production or output, denoted by  $F$ , as a function of the average capital of the representative dynasty, which we denote by  $k$ . First, we assume the usual hypothesis of monotonicity, regularity and unboundedness about the production, that is to say:  $F$  is strictly increasing and continuously differentiable from  $\mathbb{R}$  to  $\mathbb{R}$ , and

$$F(0) = 0, \quad \lim_{k \rightarrow +\infty} F(k) = +\infty$$

where we may assume  $F(x) < 0$  for every  $x \in (-\infty, 0)$ , since the assumption that  $F$  is defined in  $(-\infty, 0)$  is merely technical, as we will see later; this way we distinguish the “admissible” values of the production function from the ones which are not.

Next, we make some specific requirements. As we want to deal with a non-monotonic marginal product of capital, we assume that, in  $[0, +\infty)$ ,  $F$  is first strictly concave, then strictly convex and then again strictly concave up to  $+\infty$ . This means that in the first phase of capital accumulation, the production shows decreasing returns to scale, which become increasing from a certain level of *pro capite* capital  $\underline{k}$ . Then, when *pro capite* endowment exceed a threshold  $\bar{k} > \underline{k}$ , decreasing returns to scale characterize the production anew.

Moreover, we ask that the marginal product in  $+\infty$  is strictly positive, so that we can deal with endogenous growth. Observe that this limit surely exists, as  $F'$  is (strictly) decreasing in a neighbourhood of  $+\infty$ . Of course the assumption is equivalent to the fact that the average product of capital tends to a strictly positive quantity for large values of the average stock of capital. Moreover, requiring that the marginal product has a strictly positive lower bound is necessary to ensure a positive long-run growth rate.

As far as the agent’s behaviour is concerned, the following constraints must be satisfied, for every time  $t \geq 0$ :

$$\begin{aligned} k(t) &\geq 0, \quad c(t) \geq 0 \\ i(t) + c(t) &\leq F(k(t)), \quad \dot{k}(t) = i(t) \end{aligned}$$

where  $i(t)$  is the per capita investment at time  $t$ . Observe that the first assumption is needed in order to make the agent’s optimal strategy possibly different from the case of monotonic marginal product. In fact if condition  $\forall t \geq 0 : k(t) \geq 0$  was not present, then heuristically the convex range of production function would be not relevant to establish the long-run behaviour of economy, since every agent would have the possibility to get an amount of resources such that he can fully exploit the increasing return; therefore only the form of production function for large  $k$  would be relevant.

Another heuristic remark turns out to be crucial: the monotonicity of  $u$  respect to  $c$  implies that, if  $c$  is an optimal consumption path, then the production is completely allocated between investment and consumption, that is to say  $i(t) + c(t) = F(k(t))$  for every  $t \geq 0$ . This remark, combined with the last of the above conditions implies that the dynamics of capital allocation, for an initial endowment  $k_0 \geq 0$ , is described by the following Cauchy’s problem:

$$\begin{cases} \dot{k}(t) = F(k(t)) - c(t) & \text{for } t \geq 0 \\ k(0) = k_0 \end{cases} \quad (2)$$

Considering the first two constraints, the agent’s target can be expressed the following way: given an initial endowment of capital  $k_0 \geq 0$ , maximize the functional in (1), when  $c(\cdot)$  varies among measurable functions which are everywhere positive in  $[0, +\infty)$  and such that the unique solution to problem (2) is also everywhere positive in  $[0, +\infty)$ ; the latter requirement is usually called a *state constraint*.

A few reflections are still necessary in order to begin the analytic work. First, we will consider only the case when the time discount rate  $\hat{\rho}$  and the population growth rate  $n$  satisfy

$$\hat{\rho} - n > 0,$$

which is the most interesting from the economic point of view. Second, we weaken the requirement that  $c$  is measurable and positive in  $[0, +\infty)$  (in order that  $c$  is admissible) to the requirement that  $c$  is locally integrable and almost everywhere positive in  $[0, +\infty)$ .

Finally, we need another assumption about instantaneous utility  $u$  so that the functional in (1) is finite. To identify the best hypothesis, we temporarily restrict our attention to the particular but significant case in which  $u$  is a concave power function and  $F$  is linear; namely:

$$\begin{aligned} u(c) &= c^{1-\sigma}, \quad c \geq 0 \\ F(k) &= Lk, \quad k \geq 0 \end{aligned}$$

for some  $\sigma \in (0, 1)$  and  $L > 0$  (of course in this case  $F$  does not satisfy all of the previous assumptions). Using Gronwall's Lemma, it is easy to verify that for any admissible control  $c$  (starting from an initial state  $k_0$ ) and for every time  $t \geq 0$ ,  $\int_0^t c(s) ds \leq k_0 e^{Lt}$ . Hence, setting  $\rho = \hat{\rho} - n$ :

$$\begin{aligned} U(c(\cdot)) &= \lim_{T \rightarrow +\infty} \int_0^T e^{-\rho t} u(c(t)) dt \\ &= \lim_{T \rightarrow +\infty} e^{-\rho T} \int_0^T u(c(s)) ds + \lim_{T \rightarrow +\infty} \rho \int_0^T e^{-\rho t} \int_0^t u(c(s)) ds dt. \end{aligned}$$

Hence using Jensen inequality, we reduce the problem of the convergence of  $U(c(\cdot))$  to the problem of the convergence of

$$\int_1^{+\infty} t e^{-\rho t} e^{L(1-\sigma)t} dt$$

which is equivalent to the condition  $L(1-\sigma) < \rho$ . Perturbing this clause by the addition of a positive quantity  $\epsilon_0$  we get  $(L + \epsilon_0)(1-\sigma) < \rho - \epsilon_0$  which is in its turn equivalent to the requirement that the function  $e^{\epsilon_0 t} e^{-\rho t} (e^{(L+\epsilon_0)t})^{1-\sigma} = e^{\epsilon_0 t} e^{-\rho t} u(e^{(L+\epsilon_0)t})$  tends to 0 as  $t \rightarrow +\infty$ .

Turning back to the general case, we are suggested to assume precisely the same condition, taking care of defining the constant  $L$  as  $\lim_{k \rightarrow +\infty} F'(k)$  (which has already been assumed to be strictly positive).

### 2.3 Quantitative description

Hence the mathematical frame of the economic problem can be defined precisely as follows:

**Definition 1.** For every  $k_0 \geq 0$  and for every  $c \in \mathcal{L}_{loc}^1([0, +\infty), \mathbb{R})$ :

$k(\cdot; k_0, c)$  is the only solution to the Cauchy's problem

$$\begin{cases} k(0) = k_0 \\ \dot{k}(t) = F(k(t)) - c(t) \quad t \geq 0 \end{cases} \quad (3)$$

in the unknown  $k$ , where  $F: \mathbb{R} \rightarrow \mathbb{R}$  has the following properties:

$$F \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}), \quad F' > 0 \text{ in } \mathbb{R}, \quad F(0) = 0, \quad \lim_{x \rightarrow +\infty} F(x) = +\infty, \quad \lim_{x \rightarrow +\infty} F'(x) > 0,$$

$$F \text{ is concave in } [0, \underline{k}] \cup [\bar{k}, +\infty) \text{ for some } 0 < \underline{k} < \bar{k} \text{ and } F \text{ is convex over } [\underline{k}, \bar{k}]$$

Moreover, we set  $L := \lim_{x \rightarrow +\infty} F'(x)$ .

**Definition 2.** Let  $k_0 \geq 0$ .

The set of *admissible consumption strategies* with initial capital  $k_0$  is

$$\Lambda(k_0) := \{c \in \mathcal{L}_{loc}^1([0, +\infty), \mathbb{R}) / c \geq 0 \text{ almost everywhere, } k(\cdot; k_0, c) \geq 0\}$$

The *intertemporal utility functional*  $U(\cdot; k_0): \Lambda(k_0) \rightarrow \mathbb{R}$  is

$$U(c; k_0) := \int_0^{+\infty} e^{-\rho t} u(c(t)) dt \quad \forall c \in \Lambda(k_0)$$

where  $\rho > 0$ , and the function  $u: [0, +\infty) \rightarrow \mathbb{R}$ , representing instantaneous utility, is strictly increasing and strictly concave and satisfies:

$$\begin{aligned} u &\in \mathcal{C}^2((0, +\infty), \mathbb{R}) \cap \mathcal{C}^0([0, +\infty), \mathbb{R}), \quad u(0) = 0, \quad \lim_{x \rightarrow +\infty} u(x) = +\infty \\ \lim_{x \rightarrow 0^+} u'(x) &= +\infty, \quad \lim_{x \rightarrow +\infty} u'(x) = 0 \\ \exists \epsilon_0 > 0 : \quad \lim_{t \rightarrow +\infty} e^{\epsilon_0 t} e^{-\rho t} u(e^{(L+\epsilon_0)t}) &= 0 \end{aligned} \tag{4}$$

The *value function*  $V: [0, +\infty) \rightarrow \mathbb{R}$  is

$$V(k_0) := \sup_{c \in \Lambda(k_0)} U(c; k_0) \quad \forall k_0 \geq 0$$

*Remark 3.* The last condition in (4) implies:

$$\int_0^{+\infty} e^{-\rho t} u(e^{(L+\epsilon_0)t}) dt < +\infty, \quad \int_0^{+\infty} t e^{-\rho t} u(e^{(L+\epsilon_0)t}) dt < +\infty.$$

### 3 Preliminary results

*Remark 4.* Set

$$\overline{M} := \max_{[0, +\infty)} F' = \max\{F'(0), F'(\bar{k})\}.$$

Recalling that  $F$  is strictly increasing with  $F(0) = 0$ , we see that, for any  $x, y \in [0, +\infty)$ :

$$\begin{aligned} |F(x) - F(y)| &\leq \overline{M} |x - y| \\ F(x) &\leq \overline{M} x \end{aligned}$$

In particular  $F$  is Lipschitz-continuous.

This implies that the Cauchy's problem (3) admits a unique global solution (that is to say, defined on  $[0, +\infty)$ ).

Indeed the mapping

$$\mathcal{F}(k)(t) := k_0 + \int_0^t F(k(s)) ds - \int_0^t c(s) ds$$

is a contraction on the space  $X := \left( \mathcal{C}^0 \left( \left[ 0, \frac{1}{1+M} \right] \right), \|\cdot\|_\infty \right)$ , and so admits a unique fixed point  $k(\cdot; k_0, c)$ . Considering the mapping

$$\mathcal{F}(k)(t) := k \left( \frac{1}{1+M}; k_0, c \right) + \int_{\frac{1}{1+M}}^t F(k(s)) ds - \int_{\frac{1}{1+M}}^t c(s) ds$$

on the space  $X' := \left( \mathcal{C}^0 \left( \left[ \frac{1}{1+M}, \frac{2}{1+M} \right] \right), \|\cdot\|_\infty \right)$ , one can extend  $k(\cdot; k_0, c)$  to the interval  $\left[ \frac{1}{1+M}, \frac{2}{1+M} \right]$ , and so on.

*Remark 5.* We recall that if  $k_1$  and  $k_2$  are two solutions of (3), then the function

$$h(t) := \begin{cases} \frac{F(k_1(t)) - F(k_2(t))}{k_1(t) - k_2(t)} & \text{if } k_1(t) \neq k_2(t) \\ F'(k_1(t)) & \text{if } k_1(t) = k_2(t) \end{cases}$$

is continuous in  $[0, +\infty)$ .

As a consequence, we have a well known comparison result, which in our case can be stated as follows:

Let  $k_1, k_2 \geq 0$ ,  $c_1, c_2 \in \mathcal{L}_{loc}^1([0, +\infty), \mathbb{R})$ ,  $T_0 \geq 0$  and  $T_1 \in (T_0, +\infty]$  such that  $c_1 \leq c_2$  almost everywhere in  $[T_0, T_1]$ . Then the following implications hold:

$$k(T_0; k_1, c_1) = k(T_0; k_2, c_2) \implies \forall t \in [T_0, T_1] : k(t; k_1, c_1) \geq k(t; k_2, c_2) \quad (5)$$

$$k(T_0; k_1, c_1) > k(T_0; k_2, c_2) \implies \forall t \in [T_0, T_1] : k(t; k_1, c_1) > k(t; k_2, c_2). \quad (6)$$

**Lemma 6.** *There exists a function  $g : (0, +\infty) \rightarrow (0, +\infty)$  which is convex, decreasing and such that*

$$g(x) \leq u'(x) \quad \forall x > 0.$$

*Proof.* Let

$$\begin{aligned} \Sigma_{u'} &:= \left\{ (x, y) \in (0, +\infty)^2 / y \geq u'(x) \right\} \\ K_{u'} &:= \bigcap \left\{ K \in \mathcal{P}(\mathbb{R}^2) / K = \overline{K}, K \text{ is convex}, K \supseteq \Sigma_{u'} \right\}. \end{aligned}$$

In particular  $K_{u'}$  is a closed-convex superset of  $\Sigma_{u'}$ . Observe that, for any  $x > 0$ , the function  $H_x(y) := (x, y)$  belongs to  $\mathcal{C}^0(\mathbb{R}, \mathbb{R}^2)$ , so any set of the form

$$\{y \geq 0 / (x, y) \in K_{u'}\} = H_x^{-1}(K_{u'}) \cap [0, +\infty)$$

is closed in  $\mathbb{R}$ , and consequently it has a minimum element. Now define

$$\forall x > 0 : g(x) := \min \{y \geq 0 / (x, y) \in K_{u'}\}.$$

i) This definition implies that for every  $(x, y) \in K_{u'}$ ,  $g(x) \leq y$ ; hence

$$g(x) \leq u'(x) \quad \forall x > 0$$



because for any  $x > 0$ ,  $(x, u'(x)) \in \Sigma_{u'} \subseteq K_{u'}$ .

ii) In the second place,  $g$  is convex in  $(0, +\infty)$ . Let  $x_0, x_1 > 0$  and  $\lambda \in (0, 1)$ . By definition of  $g$ ,  $(x_0, g(x_0)), (x_1, g(x_1)) \in K_{u'}$ , which is a convex set. Hence

$$(1 - \lambda)(x_0, g(x_0)) + \lambda(x_1, g(x_1)) \in K_{u'}.$$

By the first property in i), this implies

$$g((1 - \lambda)x_0 + \lambda x_1) \leq (1 - \lambda)g(x_0) + \lambda g(x_1).$$

iii)  $g$  is decreasing. Indeed, take  $0 < x_0 < x_1$ . By ii) and by definition of convexity, for every  $n \in \mathbb{N}$ :

$$g(n(x_1 - x_0) + x_0) \geq n[g(x_1) - g(x_0)] + g(x_0).$$

Hence by the assumptions on  $u$  and by i):

$$\begin{aligned} 0 &= \lim_{n \rightarrow +\infty} u'(n(x_1 - x_0) + x_0) \geq \limsup_{n \rightarrow +\infty} g(n(x_1 - x_0) + x_0) \\ &\geq \lim_{n \rightarrow +\infty} n[g(x_1) - g(x_0)] + g(x_0) \end{aligned}$$

which implies  $g(x_1) \leq g(x_0)$ .

iv) Observe that the definition of  $g$  does not exclude that  $g(x) = 0$  for some  $x > 0$ . Indeed we show that  $g > 0$  in  $(0, +\infty)$ .

Fix  $x > 0$ , and consider the closed-convex approximation of  $\Sigma_{u'}$

$$K_x := \left\{ (t, y) \in [0, x] \times [0, +\infty) / y \geq \frac{\min_{[0, x]} u'}{x} (x - t) \right\} \cup [x, +\infty) \times [0, +\infty).$$

By construction  $K_{u'} \subseteq K_x$  which implies  $(t, g(t)) \in K_x$  for any  $t > 0$ . In particular, for every  $t \in (0, x)$ :

$$g(t) \geq \frac{\min_{[0, x]} u'}{x} (x - t) > 0$$

because  $u' > 0$ . This is precisely the fact that allows us to repeat this construction for every  $x > 0$ , which ensures that  $g > 0$  in  $(0, +\infty)$ .  $\square$

*Remark 7.* The function  $h$  defined in Remark 5 satisfies

$$|h| \leq \overline{M}.$$

where  $\overline{M}$  is defined as in Remark 4.

*Remark 8.* Let  $k_0 \geq 0$  and  $c \in \Lambda(k_0)$ . Then, for every  $t \geq 0$ :

$$\begin{aligned} k(t; k_0, c) &\leq k_0 e^{\overline{M}t} \\ \int_0^t c(s) ds &\leq k_0 e^{\overline{M}t} \end{aligned}$$

Indeed, by Remark 4 and remembering that  $c \geq 0$ , we have, for every  $t \geq 0$ ,  $\dot{k}(t; k_0, c) \leq \overline{M}k(t; k_0, c)$  - which implies by (5):

$$k(t; k_0, c) \leq k_0 e^{\overline{M}t} \quad \forall t \geq 0.$$

Now integrating both sides of the state equation, again by Remark 4 and by the fact that  $k(\cdot; k_0, c) \geq 0$  we see that, for every  $t \geq 0$ :

$$\begin{aligned} \int_0^t c(s) ds &= k_0 - k(t; k_0, c) + \int_0^t F(k(s; k_0, c)) ds \\ &\leq k_0 + \overline{M} \int_0^t k(s; k_0, c) ds \\ &\leq k_0 + \overline{M}k_0 \int_0^t e^{\overline{M}s} ds = k_0 e^{\overline{M}t}. \end{aligned}$$

**Lemma 9.** *There exists a function  $N : (0, +\infty)^2 \rightarrow (0, +\infty)$ , increasing in both variables, such that:*

*for every  $(k_0, T) \in (0, +\infty)^2$  and every  $c \in \Lambda(k_0)$ , there exists a control function  $c^T \in \Lambda(k_0)$  satisfying*

$$\begin{aligned} U(c^T; k_0) &\geq U(c; k_0) \\ c^T &= c \wedge N(k_0, T) \text{ almost everywhere in } [0, T] \end{aligned}$$

*In particular,  $c^T$  is bounded above, in  $[0, T]$ , by a quantity which does not depend on the original control  $c$ , but only on  $T$  and on the initial status  $k_0$ .*

*Proof.* Let  $g$  be the function defined in Lemma 6 and  $\beta := \frac{\log(1+\overline{M})}{\overline{M}}$ . Define, for every  $(k_0, T) \in (0, +\infty)^2$ :

$$\begin{aligned} \alpha(k_0, T) &:= \beta e^{-\rho(T+\beta)} g \left[ k_0 \left( \frac{e^{\overline{M}(T+\beta)}}{\beta} + e^{\overline{M}T} \right) \right] \\ N(k_0, T) &:= \inf \left\{ \tilde{N} > 0 / \forall N \geq \tilde{N} : u'(N) < \alpha(k_0, T) \right\}. \end{aligned}$$

In the first place,  $N(k_0, T) \neq +\infty$ , because  $\alpha(k_0, T) > 0$  for every  $k_0 > 0$ ,  $T > 0$  and  $\lim_{N \rightarrow +\infty} u'(N) = 0$ .

In the second place,  $u'((0, +\infty)) = (0, +\infty)$ , which implies  $N(k_0, T) > 0$ : otherwise, since  $(u')^{-1}(\alpha(k_0, T)) > 0$ , there would exist  $N > 0$  such that

$$\begin{aligned} N &< (u')^{-1}(\alpha(k_0, T)) \\ u'(N) &< \alpha(k_0, T) \end{aligned}$$

which is absurd because  $u'$  is decreasing; hence the quantity  $u'(N(k_0, T))$  is well defined. Moreover by the continuity of  $u'$ ,

$$u'(N(k_0, T)) \leq \alpha(k_0, T). \quad (7)$$

The function  $N(\cdot, \cdot)$  is also increasing in both variables, because  $\alpha(\cdot, \cdot)$  is decreasing in both variables and  $u'$  is decreasing.

Indeed, for  $k_0 \leq k_1$  and for a fixed  $T > 0$ , suppose that  $N(k_1, T) < N(k_0, T)$ . Then by definition of infimum we could choose  $\tilde{N} \in [N(k_1, T), N(k_0, T))$  such that  $u'(\tilde{N}) < \alpha(k_1, T)$ , which implies

$$u'(\tilde{N}) < \alpha(k_0, T)$$

by the monotonicity of  $\alpha$ . But since  $\tilde{N} > 0$ ,  $\tilde{N} < N(k_0, T)$  there also exists  $N \geq \tilde{N}$  such that  $u'(N) \geq \alpha(k_0, T)$  which implies, by the monotonicity of  $u'$ ,

$$u'(\tilde{N}) \geq \alpha(k_0, T),$$

a contradiction. With an analogous argument we prove that  $N(\cdot, \cdot)$  is increasing in the second variable.

Now let  $k_0, T > 0$  and  $c \in \Lambda(k_0)$  as in the hypothesis. If  $c \leq N(k_0, T)$  almost everywhere in  $[0, T]$ , then define  $c^T := c$ . If, on the contrary,  $c > N(k_0, T)$  in a non-negligible subset of  $[0, T]$ , then define:

$$c^T(t) := \begin{cases} c(t) \wedge N(k_0, T) & \text{if } t \in [0, T] \\ c(t) + I_T & \text{if } t \in (T, T + \beta] \\ c(t) & \text{if } t > T + \beta \end{cases}$$

where  $I_T := \int_0^T e^{-\rho t} (c(t) - c(t) \wedge N(k_0, T)) dt$ . Observe that by Remark 8:

$$\begin{aligned} 0 < I_T &\leq \int_0^T (c(t) - c(t) \wedge N(k_0, T)) dt \\ &\leq \int_0^T c(t) dt \\ &\leq k_0 e^{\overline{M}T} \end{aligned} \tag{8}$$

In order to prove the admissibility of such control function, we compare the orbit  $k := k(\cdot; k_0, c)$  to the orbit  $k^T := k(\cdot; k_0, c^T)$ . In the first place, observe that by (5) and by definition of  $c^T$ :

$$k^T(t) \geq k(t) \quad \forall t \in [0, T] \tag{9}$$

Now by the state equation, we have:

$$\dot{k}^T - \dot{k} = F(k^T) - F(k) + c - c^T. \tag{10}$$

Set for every  $t \geq 0$ :

$$h(t) := \begin{cases} \frac{F(k^T(t)) - F(k(t))}{k^T(t) - k(t)} & \text{if } k^T(t) \neq k(t) \\ F'(k(t)) & \text{if } k^T(t) = k(t) \end{cases}$$

Hence by (10)

$$\dot{k}^T(t) - \dot{k}(t) = h(t) [k^T(t) - k(t)] + c(t) - c^T(t) \quad \forall t \geq 0.$$

By Remark 5, the function  $h$  is continuous in  $[0, +\infty)$ , so this is a typical linear equation with measurable coefficient of degree one, satisfied by  $k^T - k$ . Hence, multiplying both sides by the continuous function  $t \rightarrow \exp\left(-\int_0^t h(s) ds\right)$ , we obtain:

$$\frac{d}{dt} \left\{ [k^T(t) - k(t)] e^{-\int_0^t h(s) ds} \right\} = [c(t) - c^T(t)] e^{-\int_0^t h(s) ds} \quad \forall t \geq 0$$

which implies, integrating between 0 and any  $t \geq 0$ :

$$k^T(t) - k(t) = \int_0^t [c(s) - c^T(s)] e^{\int_s^t h ds} ds \quad (11)$$

Now observe that

$$h \leq \overline{M} \text{ in } [0, +\infty) \text{ and } h \geq 0 \text{ in } [0, T] \quad (12)$$

by (9) and the monotonicity of  $F$ . Set  $t \in (T, T + \beta]$ ; then by (11) and (12):

$$\begin{aligned} k^T(t) - k(t) &= \int_0^T [c(s) - c(s) \wedge N(k_0, T)] e^{\int_s^t h ds} - I_T \cdot \int_T^t e^{\int_s^t h ds} ds \\ &\geq \int_0^T [c(s) - c(s) \wedge N(k_0, T)] ds - I_T \cdot \int_T^t e^{\overline{M}(t-s)} ds \\ &\geq \int_0^T e^{-\rho s} [c(s) - c(s) \wedge N(k_0, T)] ds - I_T \cdot \int_T^{T+\beta} e^{\overline{M}(T+\beta-s)} ds \\ &= I_T \left( 1 - \frac{e^{\overline{M}\beta} - 1}{\overline{M}} \right) = 0 \end{aligned} \quad (13)$$

This also implies, by (5) and by definition of  $c^T$ ,

$$k^T(t) \geq k(t) \quad \forall t \geq T + \beta$$

Such inequality, together with (9) and (13), gives us the general inequality

$$k^T(t) \geq k(t) \geq 0 \quad \forall t \geq 0.$$

This implies, associated with the obvious fact that  $c^T \geq 0$  almost everywhere in  $[0, +\infty)$ , that  $c^T \in \Lambda(k_0)$ .

Now we prove the “optimality” property of  $c^T$  respect to  $c$ . By the concavity of  $u$ , and setting

$N := N(k_0, T)$  for simplicity of notation, we have:

$$\begin{aligned}
U(c; k_0) - U(c^T; k_0) &= \int_0^{+\infty} e^{-\rho t} [u(c(t)) - u(c^T(t))] dt \\
&= \int_{[0, T] \cap \{c \geq N\}} e^{-\rho t} [u(c(t)) - u(c(t) \wedge N)] dt \\
&\quad + \int_T^{T+\beta} e^{-\rho t} [u(c(t)) - u(c(t) + I_T)] dt \\
&\leq \int_{[0, T] \cap \{c \geq N\}} e^{-\rho t} u'(c(t) \wedge N) [c(t) - c(t) \wedge N] dt \\
&\quad - I_T \int_T^{T+\beta} e^{-\rho t} u'(c(t) + I_T) dt \\
&= u'(N) \int_0^T e^{-\rho t} [c(t) - c(t) \wedge N] dt \\
&\quad - I_T \int_T^{T+\beta} e^{-\rho t} u'(c(t) + I_T) dt \\
&= I_T \left[ u'(N) - \int_T^{T+\beta} e^{-\rho t} u'(c(t) + I_T) dt \right] \tag{14}
\end{aligned}$$

Now we exhibit a certain lower bound which is independent on the particular control function  $c$ . By Jensen inequality, by Lemma 6 and by (8), we have:

$$\begin{aligned}
\int_T^{T+\beta} e^{-\rho t} u'(c(t) + I_T) dt &\geq \int_T^{T+\beta} e^{-\rho t} g(c(t) + I_T) dt \\
&\geq e^{-\rho(T+\beta)} \int_T^{T+\beta} g(c(t) + I_T) dt \\
&\geq \beta e^{-\rho(T+\beta)} g\left(\frac{1}{\beta} \int_T^{T+\beta} [c(t) + I_T] dt\right) \\
&\geq \beta e^{-\rho(T+\beta)} g\left(\frac{1}{\beta} \int_0^{T+\beta} c(t) dt + I_T\right) \\
&\geq \beta e^{-\rho(T+\beta)} g\left[k_0 \left(\frac{e^{\overline{M}(T+\beta)}}{\beta} + e^{\overline{M}T}\right)\right] \\
&= \alpha(k_0, T).
\end{aligned}$$

Hence by (7) and (14):

$$\begin{aligned}
U(c; k_0) - U(c^T; k_0) &\leq I_T \left[ u'(N(k_0, T)) - \int_T^{T+\beta} e^{-\rho t} u'(c(t) + I_T) dt \right] \\
&\leq I_T [u'(N(k_0, T)) - \alpha(k_0, T)] \leq 0.
\end{aligned}$$

□

**Lemma 10.** *Let  $0 < k_0 < k_1$  and  $c \in \Lambda(k_0)$ . Then there exists a control function  $\underline{c}^{k_1-k_0} \in \Lambda(k_1)$  such that*

$$U(\underline{c}^{k_1-k_0}; k_1) - U(c; k_0) \geq u'(N(k_0, k_1 - k_0) + 1) \int_0^{k_1-k_0} e^{-\rho t} dt$$

where  $N$  is the function defined in Lemma 9.

*Proof.* Fix  $k_0, k_1$  and  $c$  as in the hypothesis and take  $c^{k_1-k_0}$  as in Lemma 9 (where it is understood that  $T = k_1 - k_0$ ). Then define:

$$\underline{c}^{k_1-k_0}(t) := \begin{cases} c^{k_1-k_0}(t) + 1 & \text{if } t \in [0, k_1 - k_0) \\ c^{k_1-k_0}(t) & \text{if } t \geq k_1 - k_0 \end{cases}$$

In the first place we prove that  $\underline{c}^{k_1-k_0} \in \Lambda(k_1)$ , showing that

$$\underline{k} := k(\cdot; k_1; \underline{c}^{k_1-k_0}) > k(\cdot; k_0, c^{k_1-k_0}) =: k \quad (15)$$

over  $(0, +\infty)$ . Suppose by contradiction that this is not true, and take  $\tau := \inf \{t > 0 / \underline{k}(t) \leq k(t)\}$ . Then by the continuity of the orbits,  $\underline{k}(\tau) \leq k(\tau)$ , which implies  $\tau > 0$ . Considering the orbits as solutions to an integral equation we have:

$$\begin{aligned} k(\tau) &= k_0 + \int_0^\tau F(k(t)) dt - \int_0^\tau c^{k_1-k_0}(t) dt \\ \underline{k}(\tau) &= k_1 + \int_0^\tau F(\underline{k}(t)) dt - \int_0^\tau c^{k_1-k_0}(t) dt - \min\{\tau, k_1 - k_0\}. \end{aligned}$$

Hence

$$\begin{aligned} 0 \geq \underline{k}(\tau) - k(\tau) &= k_1 - k_0 + \int_0^\tau [F(\underline{k}(t)) - F(k(t))] dt - \min\{\tau, k_1 - k_0\} \\ &\geq \int_0^\tau [F(\underline{k}(t)) - F(k(t))] dt \end{aligned}$$

By the definition of  $\tau$  and the strict monotonicity of  $F$ , this quantity must be strictly positive, which is absurd. Hence

$$\begin{aligned} k(\cdot; k_1; \underline{c}^{k_1-k_0}) &> k(\cdot; k_0, c^{k_1-k_0}) \geq 0 \text{ in } [0, +\infty) \\ \underline{c}^{k_1-k_0} &\geq c^{k_1-k_0} \geq 0 \text{ a.e. in } [0, +\infty) \end{aligned}$$

which implies  $\underline{c}^{k_1-k_0} \in \Lambda(k_0)$ .

In the second place, remembering the properties of  $c^{k_1-k_0}$  given by Lemma 9, we have

$$\begin{aligned} U(\underline{c}^{k_1-k_0}; k_1) - U(c; k_0) &\geq U(\underline{c}^{k_1-k_0}; k_1) - U(c^{k_1-k_0}; k_0) \\ &= \int_0^{k_1-k_0} e^{-\rho t} [u(c^{k_1-k_0}(t) + 1) - u(c^{k_1-k_0}(t))] dt \\ &\geq \int_0^{k_1-k_0} e^{-\rho t} u'(c^{k_1-k_0}(t) + 1) dt \\ &\geq u'(N(k_0, k_1 - k_0) + 1) \int_0^{k_1-k_0} e^{-\rho t} dt \end{aligned}$$

which concludes the proof.  $\square$

*Remark 11.* In the previous Lemma, the property (15) can also be proved with the “comparison technique”, like we did for the admissibility of  $c^T$  in Lemma 9.

More generally, it can be proved that

$$k(\cdot; k_1, c_H) > k(\cdot; k_0, c)$$

where  $k_1 > k_0 \geq 0$ ,  $c \in \mathcal{L}_{loc}^1([0, +\infty), \mathbb{R})$  and

$$c_H(t) := \begin{cases} c(t) + H & \text{if } t \in [0, \delta_H) \\ c(t) & \text{if } t \geq \delta_H \end{cases}$$

and  $\delta_H > 0$  satisfying  $\delta_H \cdot H \leq k_1 - k_0$ .

Indeed, set  $k_H := k(\cdot; k_1, c_H)$  and  $k := k(\cdot; k_0, c)$  and suppose by contradiction that  $-\infty < \inf \{t > 0 / k_H(t) \leq k(t)\} =: \tau$ . Then for a suitable, positive continuous function  $h : [0, +\infty) \rightarrow \mathbb{R}$ , the following equality holds:

$$k_H(\tau) - k(\tau) = e^{\int_0^\tau h} \left[ k_1 - k_0 + \int_0^\tau (c(s) - c_H(s)) e^{-\int_0^s h} ds \right].$$

Moreover  $\tau \leq \delta_H$ , because on the contrary by definition of infimum we would have  $k_H > k$  in  $[0, \delta_H]$ ; then remembering (6) and the definition of  $c_H$  we would conclude that  $k_H > k$  everywhere in  $[0, +\infty)$ , which contradicts  $\tau > -\infty$ . Then the above equality implies

$$k_H(\tau) - k(\tau) > k_1 - k_0 - H \int_0^\tau e^{-\int_0^s h} ds > k_1 - k_0 - \tau H \geq k_1 - k_0 - \delta_H H \geq 0.$$

At the same time  $k_H(\tau) \leq k(\tau)$  by the continuity of  $k_h$  and  $k$  and by definition of infimum (in fact the equality holds, again by continuity); hence we have reached the desired contradiction.

Now we state a simple characterisation of the admissible constant controls.

**Proposition 12.** *Let  $k_0, c \geq 0$ . Then*

i)  $k(\cdot; k_0, F(k_0)) \equiv k_0$

ii) *the function constantly equal to  $c$  is admissible at  $k_0$  (which we write  $c \in \Lambda(k_0)$ ) if, and only if*

$$c \in [0, F(k_0)].$$

*In particular the null function is admissible at any initial state  $k_0 \geq 0$ .*

*Proof.* i) By the uniqueness of the orbit.

ii)( $\Leftarrow$ ) In the first place, observe that  $F(k_0) \in \Lambda(k_0)$ , by i). In the second place, assume  $c \in [0, F(k_0))$  and set  $k := k(\cdot; k_0, c)$ . Hence

$$\dot{k}(0) = F(k_0) - c > 0$$

which means, by the continuity of  $\dot{k}$ , that we can find  $\delta > 0$  such that  $k$  is strictly increasing in  $[0, \delta]$ . In particular  $\dot{k}(\delta) = F(k(\delta)) - c > F(k_0) - c$  because  $F$  is strictly increasing too. By the fact that  $\dot{k}(\delta) > 0$  we see that there exists  $\hat{\delta} > \delta$  such that  $k$  is strictly increasing in  $[0, \hat{\delta}]$

- and so on. Hence  $k$  is strictly increasing in  $[0, +\infty)$  and in particular  $k \geq 0$ . This shows that  $c \in \Lambda(k_0)$ .

( $\implies$ ) Suppose that  $c > F(k_0)$  and set again  $k := k(\cdot; k_0, c)$ . Then

$$\dot{k}(0) = F(k_0) - c < 0$$

so that we can find  $\delta > 0$  such that  $k$  is strictly decreasing in  $[0, \delta]$ , and  $\dot{k}(\delta) = F(k(\delta)) - c < F(k_0) - c < 0$ . Hence one can arbitrarily extend the neighbourhood of 0 in which  $\dot{k}$  is strictly less than the strictly negative constant  $F(k_0) - c$ , which implies that

$$\lim_{t \rightarrow +\infty} k(t) = -\infty.$$

Hence  $k$  cannot be everywhere-positive and  $c \notin \Lambda(k_0)$ .  $\square$

**Corollary 13.** *The set sequence  $(\Lambda(k))_{k \geq 0}$  is strictly increasing, that is:*

$$\Lambda(k_0) \subsetneq \Lambda(k_1)$$

for every  $0 \leq k_0 < k_1$ .

*Proof.* For every  $c \in \Lambda(k_0)$ ,  $k(\cdot; k_0, c) \leq k(\cdot; k_1, c)$  by 5, which implies the second orbit being positive, and so  $c \in \Lambda(k_1)$ .

On the other hand, by Proposition 12 and by the strict monotonicity of  $F$ , the constant control  $\hat{c} \equiv F(\hat{k})$  belongs to  $\Lambda(k_1) \setminus \Lambda(k_0)$  for any  $\hat{k} \in (k_0, k_1]$ .  $\square$

## 4 Basic qualitative properties of the value function

Now we deal with the first problem one has to solve in order to develop the theory: the finiteness of the value function. We start setting a result which is analogous to the one we cleared up in Remark 7, and which also follows from a certain sublinearity property of the production function  $F$ .

*Remark 14.* Set  $M_0, \hat{M} \geq 0$  such that:

$$\begin{aligned} \forall x \geq M_0 : F(x) &\leq (L + \epsilon_0)x \\ \hat{M} &:= \max_{[0, M_0]} F. \end{aligned}$$

(which is possible because  $\lim_{x \rightarrow +\infty} \frac{F(x)}{x} = L$ ). Hence, for every  $x \geq 0$ :

$$F(x) \leq (L + \epsilon_0)x + \hat{M}$$

*Remark 15.* Since  $u$  is a concave function satisfying  $u(0) = 0$ ,  $u$  is sub-additive in  $[0, +\infty)$  and satisfies:

$$\forall x > 0 : \forall K > 1 : u(Kx) \leq Ku(x)$$



**Lemma 16.** *Let  $k_0 \geq 0$ , and  $c \in \Lambda(k_0)$ . Hence, setting  $M(k_0) := 1 + \max\{(L + \epsilon_0)k_0, \hat{M}\}$ :*

$$\begin{aligned} i) \quad & \forall t \geq 0 : \int_0^t c(s) ds \leq tM(k_0) \left[1 + e^{(L+\epsilon_0)t}\right] + \frac{M(k_0)}{L + \epsilon_0} \\ ii) \quad & \lim_{t \rightarrow +\infty} e^{-\rho t} \int_0^t u(c(s)) ds = 0 \\ iii) \quad & U(c; k_0) = \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c(s)) ds dt \leq \gamma(k_0) \end{aligned}$$

where  $\gamma(k_0)$  is a finite quantity depending on  $k_0$  and on the problem's data.

*Proof.* i) Set  $\kappa := \kappa(\cdot; k_0, c)$  and  $M(k_0)$  as in the hypotheses. Observe that, by Remark 14, for every  $x \geq 0$ :

$$F(x) \leq (L + \epsilon_0)x + M(k_0).$$

Fix  $t \geq 0$ ; by the state equation, we have for any  $s \in [0, t]$

$$\kappa(s) \leq k_0 + sM(k_0) + (L + \epsilon_0) \int_0^s \kappa(\tau) d\tau$$

which implies by Gronwall's inequality:

$$\kappa(s) \leq [k_0 + sM(k_0)] e^{(L+\epsilon_0)s} \quad \forall s \in [0, t],$$

as  $s \rightarrow k_0 + sM(k_0)$  is increasing. So

$$\begin{aligned} \int_0^t (L + \epsilon_0) \kappa(s) ds & \leq k_0(L + \epsilon_0) \int_0^t e^{(L+\epsilon_0)s} ds + M(k_0)(L + \epsilon_0) \int_0^t s \cdot e^{(L+\epsilon_0)s} ds \\ & = k_0 e^{(L+\epsilon_0)t} - k_0 + tM(k_0) e^{(L+\epsilon_0)t} - \frac{M(k_0)}{(L + \epsilon_0)} e^{(L+\epsilon_0)t} + \frac{M(k_0)}{(L + \epsilon_0)} \\ & = tM(k_0) e^{(L+\epsilon_0)t} + \left[ k_0 - \frac{M(k_0)}{(L + \epsilon_0)} \right] e^{(L+\epsilon_0)t} + \frac{M(k_0)}{(L + \epsilon_0)} - k_0 \\ & \leq tM(k_0) e^{(L+\epsilon_0)t} + \frac{M(k_0)}{(L + \epsilon_0)} - k_0 \end{aligned}$$

Hence, again by the state equation, for every  $t \geq 0$ :

$$\begin{aligned} \int_0^t c(s) ds & = k_0 - \kappa(t) + \int_0^t F(\kappa(s)) ds \\ & \leq k_0 + tM(k_0) + \int_0^t (L + \epsilon_0) \kappa(s) ds \leq tM(k_0) \left[1 + e^{(L+\epsilon_0)t}\right] + \frac{M(k_0)}{(L + \epsilon_0)}. \end{aligned}$$

which proves the first assertion.

ii) In the second place, it follows by Jensen inequality, the monotonicity of  $u$  and Remark 15, that for every  $t \geq 0$ :

$$\begin{aligned} 0 \leq e^{-\rho t} \int_0^t u(c(s)) ds & \leq te^{-\rho t} u\left(\frac{\int_0^t c(s) ds}{t}\right) \leq te^{-\rho t} u\left(M(k_0) \left[1 + e^{(L+\epsilon_0)t}\right] + \frac{M(k_0)}{t(L + \epsilon_0)}\right) \\ & \leq te^{-\rho t} \left\{ u(M(k_0)) + M(k_0) u\left(e^{(L+\epsilon_0)t}\right) + u\left(\frac{M(k_0)}{t(L + \epsilon_0)}\right) \right\}; \end{aligned}$$

observe that this quantity tends to 0 as  $t \rightarrow +\infty$ , particularly by the last condition assumed in (4) about  $u$ ; so also the second claim is proven.

Finally, integrating by parts, and using ii)

$$\begin{aligned}
U(c; k_0) &= \int_0^{+\infty} e^{-\rho t} u(c(t)) dt \\
&= \lim_{T \rightarrow +\infty} \left\{ e^{-\rho T} \int_0^T u(c(s)) ds + \rho \int_0^T e^{-\rho t} \int_0^t u(c(s)) ds dt \right\} \\
&= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c(s)) ds dt \\
&\leq \rho \int_0^{+\infty} t e^{-\rho t} \left\{ u(M(k_0)) + M(k_0) u(e^{(L+\epsilon_0)t}) + u\left(\frac{M(k_0)}{t(L+\epsilon_0)}\right) \right\} dt \\
&\leq \rho u(M(k_0)) \int_0^{+\infty} t e^{-\rho t} dt + \rho M(k_0) \int_0^{+\infty} t e^{-\rho t} u(e^{(L+\epsilon_0)t}) dt \\
&\quad + \rho u\left(\frac{M(k_0)}{L+\epsilon_0}\right) \left\{ \int_0^1 e^{-\rho t} dt + \int_1^{+\infty} t e^{-\rho t} dt \right\}
\end{aligned}$$

Now it is sufficient to observe that by Remark 3 this upper bound is finite and set it equal to  $\gamma(k_0)$ .  $\square$

Hence we have established the starting point of the theory.

**Corollary 17.** *The value function  $V : [0, +\infty) \rightarrow \mathbb{R}$  is well-definite; that is, for every  $k_0 \geq 0$ ,  $V(k_0) < +\infty$ .*

*Proof.* Take  $k_0 \geq 0$  and set  $\gamma(k_0)$  as in Lemma 16. Hence:

$$V(k_0) = \sup_{c \in \Lambda(k_0)} U(c; k_0) \leq \gamma(k_0) < +\infty.$$

$\square$

Next, we prove directly some useful asymptotic properties of the value function.

**Theorem 18 (Asymptotic properties of the value function).** *The value function  $V : [0, +\infty) \rightarrow \mathbb{R}$  satisfies:*

$$\begin{aligned}
i) \quad & \lim_{k \rightarrow +\infty} V(k) = +\infty \\
ii) \quad & \lim_{k \rightarrow +\infty} \frac{V(k)}{k} = 0 \\
iii) \quad & \lim_{k \rightarrow 0} V(k) = V(0) = 0
\end{aligned}$$

*Proof.* i) For every  $k_0 \geq 0$  the constant control  $F(k_0)$  is admissible at  $k_0$  by Proposition 12; hence

$$V(k_0) \geq U(F(k_0); k_0) = \frac{u(F(k_0))}{\rho} \rightarrow +\infty$$

as  $k_0 \rightarrow +\infty$ , by the assumptions on  $u$  and  $F$ .

ii) Set  $\hat{M} > 0$  as in Remark 14 and  $k_0 > 0$  such that:

$$k_0 > \frac{1}{L + \epsilon_0} \hat{M} \quad (16)$$

Hence, for every  $x > 0$ :

$$F(x) \leq (L + \epsilon_0)(x + k_0) \quad (17)$$

By reasons that will be clear later, suppose also that:

$$k_0 > \frac{1}{L + \epsilon_0} \quad (18)$$

Observe that the proof of Lemma 16, i) does not require  $M(k_0) \geq 1$ , but only  $M(k_0) \geq \hat{M}$ ; hence (16) and (17) imply that the property in Lemma 16, i) holds for  $M(k_0) = k_0(L + \epsilon_0)$  - which means that:

$$\forall t \geq 0 : \int_0^t c(s) ds \leq k_0 + tk_0(L + \epsilon_0) \left[ 1 + e^{(L + \epsilon_0)t} \right]. \quad (19)$$

In particular

$$\forall t \geq 1 : \frac{\int_0^t c(s) ds}{t} \leq k_0 + k_0(L + \epsilon_0) + k_0(L + \epsilon_0)e^{(L + \epsilon_0)t}. \quad (20)$$

Now set

$$J_c(\alpha, \beta) := \int_\alpha^\beta te^{-\rho t} u \left( \frac{\int_0^t c(s) ds}{t} \right) dt \quad (21)$$

and fix  $N > 0$ .

We provide three different estimates, over  $J_c(0, 1)$ ,  $J_c(1, N)$  and  $J_c(N, +\infty)$ , using Remark 15.

First, we have by (19):

$$\begin{aligned} J_c(0, 1) &\leq \int_0^1 te^{-\rho t} \frac{1}{t} u \left( \int_0^1 c(s) ds \right) dt \\ &\leq u \left[ k_0 \left( 1 + (L + \epsilon_0) \left( 1 + e^{(L + \epsilon_0)} \right) \right) \right] \frac{1 - e^{-\rho}}{\rho} \\ &\leq u(k_0) \frac{1 - e^{-\rho}}{\rho} \left[ 1 + (L + \epsilon_0) \left( 1 + e^{(L + \epsilon_0)} \right) \right]. \end{aligned}$$

Moreover, by (20):

$$\begin{aligned} J_c(1, N) &\leq \int_1^N te^{-\rho t} u \left( k_0 + k_0(L + \epsilon_0) + k_0(L + \epsilon_0)e^{(L + \epsilon_0)t} \right) dt \\ &\leq u(k_0 + k_0(L + \epsilon_0)) \int_1^N te^{-\rho t} dt + u(k_0(L + \epsilon_0)) \int_1^N te^{-\rho t} e^{(L + \epsilon_0)t} dt \\ &\leq u[k_0(1 + L + \epsilon_0)] \left( 1 + e^{(L + \epsilon_0)N} \right) \int_1^N te^{-\rho t} dt \end{aligned}$$

Finally, remembering that  $k_0(L + \epsilon_0) > 1$  by (18),

$$\begin{aligned} J_c(N, +\infty) &\leq \int_N^{+\infty} te^{-\rho t} u\left(k_0 + k_0(L + \epsilon_0) + k_0(L + \epsilon_0)e^{(L+\epsilon_0)t}\right) dt \\ &\leq u(k_0 + k_0(L + \epsilon_0)) \int_N^{+\infty} te^{-\rho t} dt + k_0(L + \epsilon_0) \int_N^{+\infty} te^{-\rho t} u\left(e^{(L+\epsilon_0)t}\right) dt \end{aligned}$$

Now we show that

$$\lim_{k \rightarrow +\infty} \frac{V(k)}{k} = 0.$$

Fix  $\eta > 0$ ; by Remark 3, we can chose  $N_\eta > 0$  such that

$$(L + \epsilon_0) \int_{N_\eta}^{+\infty} te^{-\rho t} u\left(e^{(L+\epsilon_0)t}\right) dt < \eta.$$

Hence for  $k_0$  satisfying:

$$k_0 > \max \left\{ \frac{1}{L + \epsilon_0} \hat{M}, \frac{1}{L + \epsilon_0} \right\}$$

and for every  $c \in \Lambda(k_0)$ , the above estimates imply:

$$\begin{aligned} U(c; k_0) &= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c(s)) ds dt \\ &\leq \rho J_c(0, 1) + \rho J_c(1, N_\eta) + \rho J_c(N_\eta, +\infty) \\ &\leq u(k_0) (1 - e^{-\rho}) \left[ 1 + (L + \epsilon_0) \left( e^{(L+\epsilon_0)} + 1 \right) \right] + \\ &\quad + u(k_0) (1 + L + \epsilon_0) \left( 1 + e^{(L+\epsilon_0)N_\eta} \right) \int_1^{N_\eta} te^{-\rho t} dt + \\ &\quad + u(k_0) (1 + L + \epsilon_0) \int_{N_\eta}^{+\infty} te^{-\rho t} dt + k_0 \eta \end{aligned} \tag{22}$$

following Remark 15, Lemma 16, iii), (21) and Jensen inequality. Now observe that:

$$\lim_{k_0 \rightarrow +\infty} \frac{u(k_0)}{k_0} = \lim_{k_0 \rightarrow +\infty} u'(k_0) = 0.$$

Hence for  $k_0$  sufficiently large (say  $k_0 > k^*$ ):

$$\begin{aligned} \frac{u(k_0)}{k_0} &< \eta \left\{ (1 - e^{-\rho}) \left[ 1 + (L + \epsilon_0) \left( e^{(L+\epsilon_0)} + 1 \right) \right] + \right. \\ &\quad \left. + (1 + L + \epsilon_0) \left( 1 + e^{(L+\epsilon_0)N_\eta} \right) \int_1^{N_\eta} te^{-\rho t} dt + (1 + L + \epsilon_0) \int_{N_\eta}^{+\infty} te^{-\rho t} dt \right\}^{-1} \end{aligned}$$

Observe that this is possible because the expression into the brackets does not depend on  $k_0$ . In fact, like  $N_\eta$ , it depends only on  $\eta$  and on the problem's data  $L, \epsilon_0, \rho$  - and so does  $k^*$ .

By (22), this implies for every  $c \in \Lambda(k_0)$ :

$$U(c; k_0) \leq 2k_0\eta$$

which gives, taking the sup over  $\Lambda(k_0)$ :

$$V(k_0) \leq 2k_0\eta.$$

Hence the assertion is proven, because the previous inequality holds for every

$$k_0 > \max \left\{ \frac{1}{L + \epsilon_0} \hat{M}, \frac{1}{L + \epsilon_0}, k^* \right\},$$

and the last quantity is a threshold depending only on  $\eta$  and on the problem's data.

iii) In the first place, we prove that

$$V(0) = 0.$$

Let  $c \in \Lambda(0)$ ; by definition,  $c \geq 0$  so that

$$\forall t \geq 0 : \dot{k}(t; 0, c) \leq F(k(t; 0, c)).$$

Observe that  $F$  is precisely the function which defines the dynamics of  $k(\cdot; 0, 0)$ , hence by (5):

$$\forall t \geq 0 : k(t; 0, c) \leq k(t; 0, 0) = 0$$

where the last equality holds by Lemma 12, i).

Hence  $k(\cdot; 0, c) \equiv 0$  which together with  $F(0) = 0$  implies  $c \equiv 0$ . So  $\Lambda(0) = \{0\}$ , which implies

$$V(0) = U(0; 0) = \int_0^{+\infty} e^{-\rho t} u(0) dt = 0$$

Now we show that

$$\lim_{k \rightarrow 0} V(k) = 0.$$

In this case we have to study the behaviour of  $V(k_0)$  when  $k_0 \rightarrow 0$ , so we use the sublinearity of  $F(x)$  for  $x \rightarrow +\infty$  and the concavity of  $F$  near 0.

As a first step, we construct a linear function which is always above  $F$  with these two tools. Indeed we show that there is  $m > 0$  such that the function

$$G(x) := \begin{cases} mx & \text{if } x \in [0, \bar{k}] \\ (L + \epsilon_0)(x - \bar{k}) + m\bar{k} & \text{if } x \geq \bar{k} \end{cases}$$

satisfies

$$\forall x \geq 0 : F(x) \leq G(x). \quad (23)$$

If  $F'(\bar{k}) \leq L + \epsilon_0$  then it is enough to take  $m > \max \left\{ F'(0), F'(\bar{k}), \frac{F(\bar{k})}{\bar{k}} \right\}$ .

If  $F'(\bar{k}) > L + \epsilon_0$  then observe that for every  $x \geq \bar{k}$ :

$$\frac{F(x) - F(\bar{k})}{x - \bar{k}} \leq F'(x) \rightarrow L, \text{ for } x \rightarrow +\infty.$$

Hence there exists  $M \geq \bar{k}$  such that, for every  $x \geq M$ ,

$$F(x) \leq F(\bar{k}) + (L + \epsilon_0)(x - \bar{k})$$

which implies that for every  $x \geq \bar{k}$ :

$$F(x) \leq (L + \epsilon_0)(x - \bar{k}) + F(\bar{k}) + \max_{[\bar{k}, M]} F.$$

Hence we replace the third of the previous conditions on  $m$  with  $m\bar{k} > F(\bar{k}) + \max_{[\bar{k}, M]} F$ . Observe that condition  $m > F'(\bar{k})$  is still necessary to ensure that  $mx > F(x)$  for  $x \in [\underline{k}, \bar{k}]$  (Lagrange's theorem proves that it is sufficient).

Suppose also, for reasons that will be clear later, that

$$m > 1. \tag{24}$$

Now take  $k_0 > 0$ ,  $c \in \Lambda(k_0)$  and consider the function  $h : [0, +\infty) \rightarrow \mathbb{R}$  which is the unique solution to the Cauchy's problem

$$\begin{cases} h(0) = k_0 \\ \dot{h}(t) = G(h(t)) \quad t \geq 0 \end{cases}$$

Hence, by (23) and (5),  $k := k(\cdot; k_0, c) \leq h$ . So, setting

$$\bar{t} := \frac{1}{m} \log\left(\frac{\bar{k}}{k_0}\right) \text{ and } \hat{k} := \bar{k}(m - L - \epsilon_0)$$

we get, for every  $t \in [0, \bar{t}]$ :

$$h(t) = k_0 e^{mt}$$

and, for every  $t \geq \bar{t}$ :

$$\begin{aligned} h(t) &= e^{(L+\epsilon_0)t} \int_{\bar{t}}^t e^{-(L+\epsilon_0)s} \hat{k} ds + \bar{k} e^{-(L+\epsilon_0)\bar{t}} = \frac{\hat{k} e^{-(L+\epsilon_0)\bar{t}}}{L + \epsilon_0} e^{(L+\epsilon_0)t} + \bar{k} e^{-(L+\epsilon_0)\bar{t}} - \frac{\hat{k}}{L + \epsilon_0} \\ &=: \omega_0(k_0) e^{(L+\epsilon_0)t} + \omega_1(k_0) - \frac{\hat{k}}{L + \epsilon_0} \end{aligned}$$

where by definition of  $\bar{t}$  the functions  $\omega_i$  satisfy:

$$\begin{aligned} \omega_0(k_0) &= \frac{\hat{k}}{L + \epsilon_0} e^{-(L+\epsilon_0)\bar{t}} = \frac{\hat{k}}{L + \epsilon_0} \left(\frac{k_0}{\bar{k}}\right)^{\frac{L+\epsilon_0}{m}} \\ \omega_1(k_0) &= \bar{k} e^{-(L+\epsilon_0)\bar{t}} = \bar{k} \left(\frac{k_0}{\bar{k}}\right)^{\frac{L+\epsilon_0}{m}}. \end{aligned}$$

Using the state equation, we deduce two estimates for the integrals of  $c$  by the above computations of  $h$ .

For every  $t \in [0, \bar{t}]$  (remembering that  $h$  is increasing so that  $\forall s \leq t : h(s) \leq \bar{k}$ ):

$$\begin{aligned} \int_0^t c(s) \, ds &\leq k_0 + \int_0^t F(k(s)) \, ds \leq k_0 + \int_0^t G(h(s)) \, ds \\ &= k_0 + \int_0^t m k_0 e^{ms} \, ds = k_0 e^{mt}. \end{aligned} \quad (25)$$

Instead, for every  $t > \bar{t}$ :

$$\begin{aligned} \int_0^t c(s) \, ds &\leq k_0 + \int_0^{\bar{t}} G(h(s)) \, ds + \int_{\bar{t}}^t G(h(s)) \, ds \\ &\leq k_0 e^{m\bar{t}} + \int_{\bar{t}}^t \left\{ (L + \epsilon_0) h(s) + \hat{k} \right\} \, ds \\ &\leq \bar{k} + (t - \bar{t}) \hat{k} + (L + \epsilon_0) \int_{\bar{t}}^t \left\{ \omega_0(k_0) e^{(L+\epsilon_0)s} + \omega_1(k_0) - \frac{\hat{k}}{L + \epsilon_0} \right\} \, ds \\ &\leq \bar{k} + \omega_0(k_0) \left[ e^{(L+\epsilon_0)t} - e^{(L+\epsilon_0)\bar{t}} \right] + (L + \epsilon_0) (t - \bar{t}) \omega_1(k_0) \\ &\leq \bar{k} + \omega_0(k_0) e^{(L+\epsilon_0)t} - \frac{\hat{k}}{L + \epsilon_0} + (L + \epsilon_0) (t - \bar{t}) \omega_1(k_0) \end{aligned} \quad (26)$$

where we have used  $h(s) \geq \bar{k}$  for  $s \in (\bar{t}, t)$  and the fact that  $k_0 e^{m\bar{t}} = \bar{k}$ .

Now observe that

$$\begin{aligned} \lim_{k_0 \rightarrow 0} \omega_0(k_0) &= \lim_{k_0 \rightarrow 0} \omega_1(k_0) = 0 \\ \lim_{k_0 \rightarrow 0} \bar{t} &= \lim_{k_0 \rightarrow 0} \frac{1}{m} \log \left( \frac{\bar{k}}{k_0} \right) = +\infty. \end{aligned} \quad (27)$$

Hence if  $k_0$  is small enough (say  $k_0 < k^*$ ), we may assume  $\bar{t} > 1$  and  $\omega_i(k_0) \leq 1$  for  $i = 0, 1$ , so that (26) implies, for every  $t > \bar{t}$ :

$$\frac{\int_0^t c(s) \, ds}{t} \leq \bar{k} + e^{(L+\epsilon_0)t} + (L + \epsilon_0) \frac{(t - \bar{t})}{t} \leq \bar{k} + e^{(L+\epsilon_0)t} + (L + \epsilon_0) \quad (28)$$

Hence, by Lemma 16, iii) , by Remark 15, and by (25), (28), the following inequality holds for

every  $k_0 < k^*$  and every  $c \in \Lambda(k_0)$ :

$$\begin{aligned}
0 \leq U(c; k_0) &= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c(s)) ds dt \leq \rho \int_0^{+\infty} t e^{-\rho t} u\left(\frac{\int_0^t c(s) ds}{t}\right) dt \\
&\leq \rho \int_0^1 e^{-\rho t} u\left(\int_0^t c(s) ds\right) dt + \rho \int_1^{\bar{t}} t e^{-\rho t} u\left(\frac{k_0 e^{m\bar{t}}}{t}\right) dt + \\
&\quad + \rho \int_{\bar{t}}^{+\infty} t e^{-\rho t} u\left(\bar{k} + e^{(L+\epsilon_0)t} + (L+\epsilon_0)\right) dt \\
&\leq \rho \int_0^1 e^{-\rho t} u(k_0 e^{mt}) dt + \rho u\left(\frac{k_0 e^{m\bar{t}}}{\bar{t}}\right) \int_1^{\bar{t}} t e^{-\rho t} dt + \\
&\quad + \rho u(\bar{k} + (L+\epsilon_0)) \int_{\bar{t}}^{+\infty} t e^{-\rho t} dt + \rho \int_{\bar{t}}^{+\infty} t e^{-\rho t} u\left(e^{(L+\epsilon_0)t}\right) dt \\
&\leq \rho u(k_0 e^m) \int_0^1 e^{-\rho t} dt + \rho u\left(\frac{\bar{k}}{\bar{t}}\right) \frac{e^{-\rho}(1+\rho)}{\rho^2} + \\
&\quad + \rho u(\bar{k} + (L+\epsilon_0)) \int_{\bar{t}}^{+\infty} t e^{-\rho t} dt + \rho \int_{\bar{t}}^{+\infty} t e^{-\rho t} u\left(e^{(L+\epsilon_0)t}\right) dt
\end{aligned}$$

where we used also the fact that the function  $t \rightarrow \frac{e^{mt}}{t}$  is increasing for  $t > 1$ , by condition (24).

It follows from (27) and the fact that  $\lim_{x \rightarrow 0} u(x) = 0$ , together with Remark 3, that the above quantity tends to 0 as  $k_0 \rightarrow 0$ ; moreover, that quantity does not depend on  $c$ .

Hence, noticing that  $k^*$  depends only on the data and  $m$ , we see that for any  $\epsilon > 0$  there exists  $\delta \in (0, k^*]$  such that for every  $k_0 \in (0, \delta)$  and for every  $c \in \Lambda(k_0)$ :

$$U(c; k_0) \leq \epsilon,$$

which implies, taking the sup over  $\Lambda(k_0)$ , that  $V(k_0) \leq \epsilon$  - and the assertion follows.  $\square$

## 5 Existence of the optimal control

In this section we deal with a fundamental topic of any optimization problem: the existence of an optimal control. For any fixed  $k_0 \geq 0$ , we look for a control  $c^* \in \Lambda(k_0)$  satisfying

$$U(c^*; k_0) = \sup_{c \in \Lambda(k_0)} U(c; k_0) = V(k_0).$$

We preliminary observe that the peculiar features of our problem, particularly the absence of any boundedness conditions on the admissible controls, force us to make use of this result in proving certain properties of the value function which usually do not require such a settlement - and which we postpone for this reason.

First observe that by Theorem 18, iii) if we set  $c_0 \equiv 0$ , then  $U(c_0, 0) = 0 = V(0)$ ; hence  $c_0$  is optimal at 0.

Let  $k_0 > 0$ ; this will be the initial state which we will refer to during the whole section - hence the meaning of this symbol will not change in this context.

We split the construction in various steps; first we make a simple but important



*Remark 19.* Suppose that  $(f_n)_{n \in \mathbb{N}}, f$  are functions in  $\mathcal{L}_{loc}^1([0, +\infty), \mathbb{R})$  such that for every  $N \in \mathbb{N}$ ,  $f_n \rightharpoonup f$  in  $L^1([0, N], \mathbb{R})$ . If  $T > 0$ ,  $T \in \mathbb{R}$ , then it follows from the definition of weak convergence that, for  $g \in L^\infty([0, T], \mathbb{R})$ :

$$\int_0^T g(s) f_n(s) ds = \int_0^{[T]+1} \chi_{[0, T]}(s) g(s) f_n(s) ds \rightarrow \int_0^{[T]+1} \chi_{[0, T]}(s) g(s) f(s) ds = \int_0^T g(s) f(s) ds.$$

Hence  $f_n \rightharpoonup f$  in  $L^1([0, T], \mathbb{R})$ , for every  $T > 0, T \in \mathbb{R}$ .

**Step 1.** The first step is to find a maximizing sequence of controls which are admissible at  $k_0$  and a function  $\gamma \in \mathcal{L}_{loc}^1([0, +\infty), \mathbb{R})$ , such that the sequence weakly converges to  $\gamma$  in  $L^1([0, T], \mathbb{R})$ , for every  $T > 0$ .

By definition of supremum, we can find a maximizing sequence; that is to say, there exist a sequence  $(c_n)_{n \in \mathbb{N}} \subseteq \Lambda(k_0)$  of admissible controls satisfying:

$$\lim_{n \rightarrow +\infty} U(c_n; k_0) = V(k_0).$$

In order to apply the tools we set up at the beginning of the chapter, we need the following result.

**Lemma 20.** Let  $T \in \mathbb{N}$  and  $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{L}_{loc}^1([0, +\infty), \mathbb{R})$ ,  $M(T) > 0$  such that

$$\forall n \in \mathbb{N} : \|f_n\|_{\infty, [0, T]} \leq M(T).$$

Then there exist a subsequence  $(\bar{f}_n)_{n \in \mathbb{N}}$  of  $(f_n)_{n \in \mathbb{N}}$  and a function  $f \in L^1([0, T], \mathbb{R})$  such that

$$\bar{f}_n \rightharpoonup f \text{ in } L^1([0, T], \mathbb{R}).$$

*Proof.* For every  $0 \leq t_0 < t_1 \leq T$ :

$$\int_{t_0}^{t_1} |f_n(s)| ds \leq \|f_n\|_{\infty, [0, T]} \cdot (t_1 - t_0) \leq M(T) \cdot (t_1 - t_0).$$

Hence, by the fact that the family  $\{(t_0, t_1) \in \mathcal{P}([0, T]) / t_0, t_1 \in [0, T]\}$  generates the Borel  $\sigma$ -algebra in  $[0, T]$ , we deduce that the latter relation holds for every measurable set  $E \subseteq [0, T]$ ; that is to say

$$\int_E |f_n(s)| ds \leq M(T) \cdot \mu(E).$$

This implies easily that the densities  $\{d_n / n \in \mathbb{N}\}$  given by  $d_n(E) := \int_E f_n(s) ds$  are absolutely equicontinuous. So the thesis follows from the Dunford-Pettis criterion (see [4]). Observe that the third condition required by such theorem, that is to say, for any  $\epsilon > 0$  there exists a compact set  $K_\epsilon \subseteq [0, T]$  such that

$$\forall n \in \mathbb{N} : \int_{[0, T] \setminus K_\epsilon} f_n(s) ds \leq \epsilon$$

is obviously satisfied. □

Now we apply Lemma 9 to  $(c_n)_{n \in \mathbb{N}}$  in order to find a new sequence  $(c_n^1)_{n \in \mathbb{N}} \subseteq \Lambda(k_0)$  such that, for every  $n \in \mathbb{N}$ :

$$\begin{aligned} U(c_n^1; k_0) &\geq U(c_n; k_0) \\ c_n^1 &= c_n \wedge N(k_0, 1) \text{ a.e. in } [0, 1]. \end{aligned}$$

In particular  $(c_n^1)_{n \in \mathbb{N}} \subseteq \mathcal{L}_{loc}^1([0, +\infty), \mathbb{R})$  and  $\|c_n^1\|_{\infty, [0, 1]} \leq N(k_0, 1)$  for every  $n \in \mathbb{N}$ . Hence by Lemma 20, there exists a sequence  $(\bar{c}_n^1)_{n \in \mathbb{N}}$  extracted from  $(c_n^1)_{n \in \mathbb{N}}$  and a function  $c^1 \in L^1([0, 1], \mathbb{R})$  such that

$$\bar{c}_n^1 \rightharpoonup c^1 \text{ in } L^1([0, 1], \mathbb{R}).$$

Now define, for every  $n \in \mathbb{N}$ :

$$c_n^2 := (\bar{c}_n^1)^2$$

where  $(\bar{c}_n^1)^2$  is understood with the notation of Lemma 9.

Hence for every  $n \in \mathbb{N}$ :

$$\begin{aligned} U(c_n^2; k_0) &\geq U(\bar{c}_n^1; k_0) \\ c_n^2 &= \bar{c}_n^1 \wedge N(k_0, 2) \text{ a.e. in } [0, 2]. \end{aligned}$$

Again by Lemma 20, we can exhibit a subsequence  $(\bar{c}_n^2)_{n \in \mathbb{N}}$  of  $(c_n^2)_{n \in \mathbb{N}}$  and a function  $c^2 \in L^1([0, 2], \mathbb{R})$  such that

$$\bar{c}_n^2 \rightharpoonup c^2 \text{ in } L^1([0, 2], \mathbb{R}).$$

Following this pattern we are able to give a recursive definition of a family

$\left\{ \left( (c_n^T)_{n \in \mathbb{N}}, (\bar{c}_n^T)_{n \in \mathbb{N}}, c^T \right) / T \in \mathbb{N} \right\}$  and  $\{i(T, n) \in [0, +\infty) / T, n \in \mathbb{N}\}$  satisfying, for every  $T, n \in \mathbb{N}$ :

$$\begin{aligned} c_n^T &\in \Lambda(k_0), \bar{c}_n^T = c_{n+i(T, n)}^T \\ U(c_n^{T+1}; k_0) &\geq U(\bar{c}_n^T; k_0) \\ c_n^{T+1} &= \bar{c}_n^T \wedge N(k_0, T+1) \text{ a.e. in } [0, T+1] \\ \bar{c}_n^T &\rightharpoonup c^T \text{ in } L^1([0, T], \mathbb{R}) \end{aligned} \tag{29}$$

Now we show that, for every  $T \in \mathbb{N}$ ,

$$c^{T+1} = c^T \text{ almost everywhere in } [0, T]. \tag{30}$$

Assume the notation “ $\tilde{\forall} s \in A : P(s)$ ” with the meaning “for almost every  $s \in A$ ,  $P(s)$  holds”. Hence:

$$\tilde{\forall} s \in [0, T] : \bar{c}_n^{T+1}(s) = c_{n+i(T, n)}^{T+1}(s) = \bar{c}_{n+i(T, n)}^T(s) \wedge N(k_0, T+1) = \bar{c}_{n+i(T, n)}^T(s)$$

where the last equality holds because, by the penultimate condition in (29) and by the monotonicity of the function  $N(k_0, \cdot)$ , for any  $p \in \mathbb{N}$ :

$$\|\bar{c}_p^T\|_{\infty, [0, T]} = \|c_{p+i(T, p)}^T\|_{\infty, [0, T]} \leq N(k_0, T) \leq N(k_0, T+1).$$

Hence the assertion in (30) follows from the essential uniqueness of the weak limit in  $L^1([0, T], \mathbb{R})$ . Now we want to set up a diagonal procedure in order to exhibit a sequence  $(\gamma_n)_{n \in \mathbb{N}} \subseteq \Lambda(k_0)$  and a function  $\gamma \in \mathcal{L}_{loc}^1([0, +\infty), \mathbb{R})$  such that

$$\gamma_n \rightharpoonup \gamma \text{ in } L^1([0, T], \mathbb{R}) \quad \forall T > 0.$$

**Definition 21.** i)  $\gamma : [0, +\infty) \rightarrow \mathbb{R}$  is the function

$$\gamma(t) := c^{[t]+1}(t) \quad \forall t \geq 0$$

ii) The sequence  $(\gamma_n)_{n \in \mathbb{N}}$  is defined as follows:

$$\begin{cases} \gamma_1 := \bar{c}_1^1 \\ \forall n \geq 2 : \text{ if } \gamma_n = c_{j(n)}^n \text{ then } \gamma_{n+1} = \bar{c}_m^{n+1}, \\ \text{where } m := \min \left\{ k \in \mathbb{N} / k > j(n) \text{ and } c_k^{n+1} \in (\bar{c}_p^{n+1})_{p \in \mathbb{N}} \right\} \end{cases}$$

This diagonal procedure is resumed by the following scheme, in which the elements of the (weakly) convergent subsequences  $(\bar{c}_n^m)_{n \in \mathbb{N}}$ ,  $m \geq 1$  are emphasized by the square brackets.

$$\begin{array}{cccccccccccccccc} c_1^1 & c_2^1 & \dots & c_h^1 & \dots & [\mathbf{c}_{j(1)}^1] & \dots & c_i^1 & \dots & c_k^2 & \dots & c_{j(2)}^1 & \dots & c_{j(3)}^1 \\ c_1^2 & c_2^2 & \dots & [c_h^2] & \dots & c_{j(1)}^2 & \dots & c_i^2 & \dots & c_k^2 & \dots & [\mathbf{c}_{j(2)}^2] & \dots & c_{j(3)}^2 \\ c_1^3 & c_2^3 & \dots & c_h^3 & \dots & c_{j(1)}^3 & \dots & [c_i^3] & \dots & [c_k^3] & \dots & c_{j(2)}^3 & \dots & [\mathbf{c}_{j(3)}^3] \\ c_1^4 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array}$$

*Remark 22.* Let  $T \in \mathbb{N}$ . Condition (30) implies that  $\gamma = c^T$  almost everywhere in  $[0, T]$ . Hence it follows from (29) that:

$$\bar{c}_n^T \rightharpoonup \gamma \text{ in } L^1([0, T], \mathbb{R}).$$

We have shown that  $(\bar{c}_n^{T+1})_{n \in \mathbb{N}}$ , restricted to  $[0, T]$ , almost coincides with a subsequence of  $(\bar{c}_n^T)_{n \in \mathbb{N}}$ ; we want to prove an analogous result in relation to  $(\gamma_n)_{n \in \mathbb{N}}$ .

We have  $\gamma_1 := \bar{c}_1^1 = c_{j(1)}^1$  (with  $j(1) = 1 + i(1, 1)$ ), so by Definition 21, there exists  $m_2 > j(1)$  such that  $\gamma_2 = \bar{c}_{m_2}^2 = c_{j(m_2)}^2$ , where  $j(m_2) := m_2 + i(2, m_2)$ . Hence

$$\forall s \in [0, 1] : \gamma_2(s) = \bar{c}_{m_2}^2(s) = c_{j(m_2)}^2(s) = \bar{c}_{j(m_2)}^1(s) \wedge N(k_0, 2) = \bar{c}_{j(m_2)}^1(s)$$

where the last equality again holds because by construction  $\|\bar{c}_p^1\|_{\infty, [0, 1]} \leq N(k_0, 1) \leq N(k_0, 2)$  for any  $p \in \mathbb{N}$ .

Moreover for some  $m_3 > j(m_2)$ ,  $\gamma_3 = \bar{c}_{m_3}^3$ ; setting  $j(m_3) := m_3 + i(3, m_3)$  and  $j(j(m_3)) := j(m_3) + i(2, j(m_3))$ , we have:

$$\begin{aligned} \forall s \in [0, 1] : \gamma_3(s) &= \bar{c}_{m_3}^3(s) = c_{j(m_3)}^3(s) = \bar{c}_{j(m_3)}^2(s) \wedge N(k_0, 3) \\ &= c_{j(j(m_3))}^2(s) \wedge N(k_0, 3) = \bar{c}_{j(j(m_3))}^1(s) \wedge N(k_0, 2) \wedge N(k_0, 3) \\ &= \bar{c}_{j(j(m_3))}^1(s) \end{aligned}$$

as  $N(k_0, 1) \leq N(k_0, 2) \leq N(k_0, 3)$ , and

$$\begin{aligned} \forall s \in [0, 2] : \gamma_3(s) &= c_{j(m_3)}^3(s) = \bar{c}_{j(m_3)}^2(s) \wedge N(k_0, 3) \\ &= \bar{c}_{j(m_3)}^2 \end{aligned}$$

Hence, by the fact that  $1 < j(m_2) < j(j(m_3))$ , we see that  $(\gamma_1, \gamma_2, \gamma_3)$  coincides with a subsequence of  $(\bar{c}_n^1)_{n \in \mathbb{N}}$  almost everywhere in  $[0, 1]$ ; it follows from  $j(m_3) > m_2$  that  $(\gamma_2, \gamma_3)$  coincides with a subsequence of  $(\bar{c}_n^2)_{n \in \mathbb{N}}$  almost everywhere in  $[0, 2]$ . Obviously this reasoning can be repeated to prove by induction the following

**Proposition 23.** *Let  $(\gamma_n)_{n \in \mathbb{N}}$ ,  $\gamma$  as in Definition 21. Then  $(\gamma_n)_{n \in \mathbb{N}} \subseteq \Lambda(k_0), \gamma \in \mathcal{L}_{loc}^1([0, +\infty), \mathbb{R})$  and*

$$\lim_{n \rightarrow +\infty} U(\gamma_n; k_0) = V(k_0).$$

Moreover, for every  $T \in \mathbb{N}$ ,  $(\gamma_n)_{n \geq T}$  coincides almost everywhere in  $[0, T]$  with a subsequence of  $(\bar{c}_n^T)_{n \in \mathbb{N}}$ . Consequently

$$\begin{aligned} \gamma_n &\rightharpoonup \gamma \text{ in } L^1([0, T], \mathbb{R}) \quad \forall T > 0, T \in \mathbb{R}, \\ \|\gamma_n\|_{\infty, [0, T]} &\leq N(k_0, T) \quad \forall T, n \in \mathbb{N}. \end{aligned}$$

*Proof.* By Remark 22, for every  $T \in \mathbb{N}$ ,  $\gamma = c^T$  almost everywhere in  $[0, T]$ ; hence  $\gamma \in L^1([0, T], \mathbb{R})$ , which implies  $\gamma \in \mathcal{L}_{loc}^1([0, +\infty), \mathbb{R})$  because  $T$  is generic.

By Definition 21,  $\gamma_1 = c_{j(1)}^1$  for some  $j(1) \geq 1$ ; hence by induction we have, for every  $n \in \mathbb{N}$ ,  $\gamma_n = \bar{c}_{j(n)}^n$  for some  $j(n) \geq n$ ; in particular, by the first condition in (29),  $\gamma_n \in \Lambda(k_0)$ . With  $n \rightarrow j(n)$  defined this way, set  $p(n) := j(n) + i(n, j(n))$ ; so remembering the other conditions in (29):

$$\begin{aligned} |U(\gamma_n; k_0) - V(k_0)| &= V(k_0) - U(\gamma_n; k_0) = V(k_0) - U(\bar{c}_{j(n)}^n; k_0) \\ &= V(k_0) - U(\bar{c}_{p(n)}^n; k_0) \leq V(k_0) - U(\bar{c}_{p(n)}^{n-1}; k_0) \\ &= V(k_0) - U(\bar{c}_{p(n)+i(n-1, p(n))}^{n-1}; k_0) \\ &\leq \dots \leq V(k_0) - U(\bar{c}_{q(n)}^1; k_0) \\ &\leq V(k_0) - U(c_{q(n)}; k_0) = |U(c_{q(n)}; k_0) - V(k_0)|, \end{aligned}$$

for some  $q(n) \geq p(n) \geq n$ . Hence the first assertion follows from the fact that  $\lim_{k \rightarrow +\infty} U(c_k; k_0) = V(k_0)$ .

Now fix  $T \in \mathbb{N}$ . The argument developed after Remark 22 inductively shows that there exists a sequence of natural numbers  $n \rightarrow k_T(n)$  such that

$$\forall n \geq T : \forall s \in [0, T] : \gamma_n(s) = \bar{c}_{n+k_T(n)}^T(s).$$

This implies by Remark 22 that  $\gamma_n \rightharpoonup \gamma$  in  $L^1([0, T], \mathbb{R})$ .

As this holds for every  $T \in \mathbb{N}$ , it is a consequence of Remark 19 that it must hold for every real number  $T > 0$ . The last condition obviously holds by construction and by (29).  $\square$

The first step is then accomplished.

**Step 2.** The next step is to show that  $\gamma$  is admissible at  $k_0$ . For this purpose, it is enough to prove the following

**Proposition 24.** *Let  $T > 0$ . Hence  $\gamma \geq 0$  almost everywhere in  $[0, T]$ , and, for every  $t \in [0, T]$ ,  $k(t; k_0, \gamma) \geq 0$ .*

*Proof.* It is well known that the weak convergence of  $(\gamma_n)_{n \in \mathbb{N}}$  to  $\gamma$  in  $L^1([0, T], \mathbb{R})$ , ensured by Proposition 23, implies that

$$\liminf_{n \rightarrow +\infty} \gamma_n(t) \leq \gamma(t) \text{ a.e. in } [0, T].$$

Moreover,  $(\gamma_n)_{n \in \mathbb{N}} \subseteq \Lambda(k_0)$ , hence any  $\gamma_n$  is almost everywhere non-negative in  $[0, T]$ . This implies  $\gamma \geq 0$  almost everywhere in  $[0, T]$ .

Set  $\kappa := k(\cdot; k_0, \gamma)$  and  $\kappa_n := k(\cdot; k_0, \gamma_n)$ ; we show that, for every  $t \in [0, T]$ :

$$\limsup_{n \rightarrow +\infty} \kappa_n(t) \leq \kappa(t).$$

Then the second assertion will follow from the fact that  $\kappa_n \geq 0$  in  $[0, T]$  for any  $n \in \mathbb{N}$ , by the admissibility of the  $\gamma_n$ 's.

Fix  $n \in \mathbb{N}$ . Subtracting the state equation for  $\kappa$  from the state equation for  $\kappa_n$ , we obtain, for every  $t \in [0, T]$ :

$$\dot{\kappa}_n(t) - \dot{\kappa}(t) = F(\kappa_n(t)) - F(\kappa(t)) - [\gamma_n(t) - \gamma(t)] \leq \overline{M}[\kappa_n(t) - \kappa(t)] - [\gamma_n(t) - \gamma(t)]$$

which implies

$$[\kappa_n(t) - \kappa(t)]e^{-\overline{M}t} - e^{-\overline{M}t}\overline{M}[\kappa_n(t) - \kappa(t)] \leq e^{-\overline{M}t}[\gamma(t) - \gamma_n(t)]$$

that is to say:

$$\frac{d}{dt} [\kappa_n(t) - \kappa(t)]e^{-\overline{M}t} \leq e^{-\overline{M}t}[\gamma(t) - \gamma_n(t)].$$

Hence, for every fixed  $t \in [0, T]$ :

$$\kappa_n(t) - \kappa(t) \leq \int_0^t e^{\overline{M}(t-s)} [\gamma(s) - \gamma_n(s)] ds = \int_0^T \chi_{[0,t]}(s) e^{\overline{M}(t-s)} [\gamma(s) - \gamma_n(s)] ds.$$

The function  $s \rightarrow \chi_{[0,t]}(s) e^{\overline{M}(t-s)}$  is bounded in  $[0, T]$  (by 1 and  $e^{\overline{M}t}$ ), hence we can apply the weak convergence  $\gamma_n \rightharpoonup \gamma$  in  $L^1([0, T], \mathbb{R})$  to deduce that the quantity at the right-hand member of the above inequality tends to 0 as  $n \rightarrow +\infty$ . Hence

$$\limsup_{n \rightarrow +\infty} \kappa_n(t) \leq \kappa(t).$$

□

As a consequence,  $\gamma$  is almost everywhere non-negative in  $[0, +\infty)$  and  $k(\cdot; k_0, \gamma)$  is everywhere non-negative in  $[0, +\infty)$  - which precisely means that  $\gamma \in \Lambda(k_0)$ . Hence the second step is also ended.

**Step 3.** Now it is time to define the control which is optimal at  $k_0$ . In order to do this, we need to extract a subsequence from  $(\gamma_n)_{n \in \mathbb{N}}$  because the weak convergence to  $\gamma$  in the intervals could not be enough to ensure that  $\lim_{n \rightarrow +\infty} U(\gamma_n; k_0) = U(\gamma; k_0)$ ; we will also need the admissibility of  $\gamma$ . By the last assertion stated in Proposition 23, and by the monotonicity of  $u$ , we have:

$$\|u(\gamma_n)\|_{\infty, [0,1]} \leq u(N(k_0, 1)) \quad \forall n \in \mathbb{N}.$$

Hence by Lemma 20, there exists a function  $f^1 \in L^1([0, 1], \mathbb{R})$  and a sequence  $(u(\gamma_{1,n}))_{n \in \mathbb{N}}$  extracted from  $(u(\gamma_n))_{n \in \mathbb{N}}$ , such that

$$u(\gamma_{1,n}) \rightharpoonup f^1 \text{ in } L^1([0, 1], \mathbb{R}).$$

Again by Proposition 23 and the monotonicity of  $u$ ,

$$\|u(\gamma_{1,n})\|_{\infty, [0,2]} \leq u(N(k_0, 2)) \quad \forall n \in \mathbb{N}$$

which implies by Lemma 20 the existence of  $f^2 \in L^1([0, 2], \mathbb{R})$  and of a sequence  $(u(\gamma_{2,n}))_{n \in \mathbb{N}}$  extracted from  $(u(\gamma_{1,n}))_{n \in \mathbb{N}}$  such that

$$u(\gamma_{2,n}) \rightharpoonup f^2 \text{ in } L^1([0, 2], \mathbb{R});$$

in particular  $f^2 = f^1$  almost everywhere in  $[0, 1]$  by the essential uniqueness of the weak limit. Going on this way we see that there exists a family  $\{(u(\gamma_{T,n}))_{n \in \mathbb{N}}, f^T) / T \in \mathbb{N}\}$  satisfying, for every  $T \in \mathbb{N}$ :

$$\begin{aligned} \|u(\gamma_{T,n})\|_{\infty, [0,T]} &\leq u(N(k_0, T)) \quad \forall n \in \mathbb{N} \\ (u(\gamma_{T+1,n}))_{n \in \mathbb{N}} &\text{ is extracted from } (u(\gamma_{T,n}))_{n \in \mathbb{N}} \\ f^{T+1} &= f^T \text{ almost everywhere in } [0, T] \\ u(\gamma_{T,n}) &\rightharpoonup f^T \text{ in } L^1([0, T], \mathbb{R}). \end{aligned}$$

Hence, for every  $T \in \mathbb{N}$ , the sequence  $(u(\gamma_{n,n}))_{n \geq T}$  is extracted from  $(u(\gamma_{T,n}))_{n \in \mathbb{N}}$ . If we define  $f(t) := f^{[t]+1}(t)$ , then  $f = f^T$  almost everywhere in  $[0, T]$ . So

$$u(\gamma_{n,n}) \rightharpoonup f \text{ in } L^1([0, T], \mathbb{R}) \quad \forall T > 0. \quad (31)$$

by construction and by Remark 19. This implies that

$$0 \leq \liminf_{n \rightarrow +\infty} u(\gamma_{n,n}(t)) \leq f(t)$$

for almost every  $t \in \mathbb{R}$ .

Now define  $c^* : [0, +\infty) \rightarrow \mathbb{R}$  as

$$c^*(t) := \begin{cases} u^{-1}(f(t)) & \text{if } f(t) \geq 0 \\ 0 & \text{if } f(t) < 0. \end{cases}$$

Obviously  $c^* \geq 0$  everywhere in  $\mathbb{R}$ . Moreover, again by the properties of the weak convergence, for any  $T \in \mathbb{N}$  and for almost every  $t \in [0, T]$ :

$$f(t) \leq \limsup_{n \rightarrow +\infty} u(\gamma_{n,n}(t)) \leq u(N(k_0, T)).$$

This implies, together with the fact that  $u^{-1}$  is increasing, that  $c^*$  is bounded above by  $N(k_0, T)$  almost everywhere in  $[0, T]$ . As this holds for every  $T \in \mathbb{N}$ ,

$$c^* \in L_{loc}^\infty([0, +\infty), \mathbb{R}). \quad (32)$$

To complete the proof of the admissibility of  $c^*$ , we show that  $c^* \leq \gamma$  almost everywhere in  $[0, +\infty)$ .

Fix  $T > 0$  and let  $t_0 \in [0, T]$  be a Lebesgue point for both  $f$  and  $\gamma$  in  $[0, T]$ ; then take  $t_1 \in (t_0, T)$ . By the concavity of  $u$  and by Jensen inequality:

$$\frac{\int_{t_0}^{t_1} u(\gamma_{n,n}(s)) ds}{t_1 - t_0} \leq u\left(\frac{\int_{t_0}^{t_1} \gamma_{n,n}(s) ds}{t_1 - t_0}\right) \quad (33)$$

Observe that  $(\gamma_{n,n})_{n \geq 1}$  is a subsequence of  $(\gamma_{1,n})_{n \in \mathbb{N}}$ , which is in its turn extracted from  $(\gamma_n)_{n \in \mathbb{N}}$ . Hence  $\gamma_{n,n} \rightharpoonup \gamma$  in  $L^1([0, T], \mathbb{R})$ , which implies  $\lim_{n \rightarrow +\infty} \int_{t_0}^{t_1} \gamma_{n,n}(s) ds = \int_{t_0}^{t_1} \gamma(s) ds$ . So taking the limit for  $n \rightarrow +\infty$  in (33), by the continuity of  $u$  and by (31), we have:

$$\frac{\int_{t_0}^{t_1} f(s) ds}{t_1 - t_0} \leq u\left(\frac{\int_{t_0}^{t_1} \gamma(s) ds}{t_1 - t_0}\right).$$

As  $t_0$  is a Lebesgue point for both  $f$  and  $\gamma$  in  $[0, T]$ , we can take the limit for  $t_1 \rightarrow t_0$  in the previous inequality and get  $f(t_0) \leq u(\gamma(t_0))$ .

By the Lebesgue Point Theorem, this argument works for almost every  $t_0 \in [0, T]$ . So by the monotonicity of  $u^{-1}$  we deduce

$$c^* \leq \gamma \text{ almost everywhere in } [0, T].$$

Because  $T$  is generic, we have by (5):  $k(t; k_0, c^*) \geq k(t; k_0, \gamma)$  for every  $t \in \mathbb{R}$ . Hence by the admissibility of  $\gamma$  at  $k_0$ ,  $k(\cdot; k_0, c^*) \geq 0$ . This implies, together with (32) and  $c^* \geq 0$  in  $[0, +\infty)$ ,

$$c^* \in \Lambda(k_0).$$

Then by Proposition 23, by the fact that  $(\gamma_{n,n})_{n \in \mathbb{N}}$  is extracted from  $(\gamma_n)_{n \in \mathbb{N}}$ , by Lemma 16, iii), by (31) and by Fatou's Lemma:

$$\begin{aligned} V(k_0) &= \lim_{n \rightarrow +\infty} U(\gamma_n; k_0) = \lim_{n \rightarrow +\infty} U(\gamma_{n,n}; k_0) \\ &= \lim_{n \rightarrow +\infty} \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(\gamma_{n,n}(s)) ds dt \\ &\leq \rho \int_0^{+\infty} e^{-\rho t} \limsup_{n \rightarrow +\infty} \int_0^t u(\gamma_{n,n}(s)) ds dt \\ &= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t f(s) ds dt \\ &= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c^*(s)) ds dt = U(c^*; k_0). \end{aligned}$$

Hence we have proved that for every  $k_0 \geq 0$  there exists  $c^* \in \Lambda(k_0)$  which is optimal at  $k_0$  and everywhere positive in  $\mathbb{R}$ , satisfying:

$$c^* \in L_{loc}^\infty([0, +\infty), \mathbb{R}).$$

## 6 Further properties of the value function

Now it is possible to set some regularity properties of the value function, with the help of optimal controls. The next theorem uses the monotonicity with respect to the first variable of the function defined in Lemma 9.

**Theorem 25.** *The value function  $V : [0, +\infty) \rightarrow \mathbb{R}$  satisfies:*

i)  *$V$  is strictly increasing*

ii) *For every  $k_0 > 0$ , there exists  $C(k_0), \delta > 0$  such that for every  $h \in (-\delta, \delta)$ :*

$$\frac{V(k_0 + h) - V(k_0)}{h} \geq C(k_0)$$

iii)  *$V$  is Lipschitz-continuous in every closed sub-interval of  $(0, +\infty)$ .*

*Proof.* i) Let  $0 < k_0 < k_1$ . Set  $c \in (0, F(k_0)]$  and  $c_0 \equiv c$  in  $[0, +\infty)$ ; hence by Lemma 12 and by Theorem 18,

$$V(0) = 0 < \frac{u(c)}{\rho} = U(c_0; k_0) \leq V(k_0).$$

In order to establish that  $V(k_0) < V(k_1)$ , take  $c \in \Lambda(k_0)$  optimal at  $k_0$  and define  $\underline{c}^{k_1-k_0}$  as in Lemma 10. As

$$u'(N(k_0, k_1 - k_0) + 1) \int_0^{k_1-k_0} e^{-\rho t} dt > 0$$

we have

$$V(k_0) = U(c; k_0) < U(\underline{c}^{k_1-k_0}; k_1) \leq V(k_1)$$

ii) We split the proof in two parts.

First, take  $k_0, h > 0$ ,  $c$  optimal at  $k_0$  and set  $k_1 := k_0 + h$ . Because  $k_1 > k_0$  we can choose  $\underline{c}^{k_1-k_0} = \underline{c}^h \in \Lambda(k_0 + h)$  as in Lemma 10. Hence

$$V(k_0 + h) - V(k_0) \geq U(\underline{c}^h; k_0 + h) - U(c; k_0) \geq u'(N(k_0, h) + 1) \int_0^h e^{-\rho t} dt$$

Now, by the fact that  $\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h e^{-\rho t} dt = 1$  and that  $N(k_0, \cdot)$  is increasing, there exists  $\delta > 0$  such that, for any  $h \in (0, \delta)$ :

$$\frac{V(k_0 + h) - V(k_0)}{h} \geq u'(N(k_0, h) + 1) \frac{\int_0^h e^{-\rho t} dt}{h} \geq \frac{u'(N(k_0, 1) + 1)}{2} =: C(k_0)$$

In the second place, fix  $k_0 > 0$ ,  $h < 0$  and  $c$  optimal at  $k_0 + h$ .



Then again take  $\underline{c}^{k_0-(k_0+h)} = \underline{c}^{-h} \in \Lambda(k_0)$  as in Lemma 10. Hence

$$\begin{aligned} V(k_0+h) - V(k_0) &\leq U(c; k_0+h) - U(\underline{c}^{-h}; k_0) \\ &\leq -u'(N(k_0+h, -h) + 1) \int_0^{-h} e^{-\rho t} dt. \end{aligned}$$

We can assume that  $-\frac{1}{h} \int_0^{-h} e^{-\rho t} dt \geq \frac{1}{2}$  for  $-\delta < h < 0$ . Hence, by the monotonicity of  $N(\cdot, \cdot)$  in both variables, for every  $h \in (-\delta, 0)$ :

$$\frac{V(k_0+h) - V(k_0)}{h} \geq \frac{u'(N(k_0+h, -h) + 1)}{2} \geq \frac{u'(N(k_0, 1) + 1)}{2} = C(k_0).$$

iii) Let  $0 < k_0 < k_1$ . We want a reverse inequality for  $V(k_1) - V(k_0)$ , so take  $c_1 \in \Lambda(k_1)$  optimal at  $k_1$ . In order to define the proper  $c_0 \in \Lambda(k_0)$ , observe that the orbit  $k = k(\cdot; k_0, 0)$  (with null control) satisfies  $\dot{k} = F(k)$ . With an argument similar to the one used in Proposition 12 we can see that  $\dot{k}(t) > F(k_0) > 0$  for every  $t > 0$ , and so  $\lim_{t \rightarrow +\infty} k(t) = +\infty$ .

Then by Darboux's property there exists  $\bar{t} > 0$  such that  $k(\bar{t}) = k_1$ . Observe that, since  $k$  and  $F$  are strictly increasing functions,  $\dot{k}$  must also be strictly increasing. Hence applying Lagrange's theorem to  $k$  gives for some  $\xi \in (0, \bar{t})$ :

$$k_1 - k_0 = k(\bar{t}) - k(0) = \bar{t} \cdot \dot{k}(\xi) > \bar{t} \dot{k}(0) = \bar{t} F(k_0) \quad (34)$$

Now define

$$c_0(t) := \begin{cases} 0 & \text{if } t \in [0, \bar{t}] \\ c_1(t - \bar{t}) & \text{if } t > \bar{t} \end{cases}$$

It is easy to check that  $c_0 \in \Lambda(k_0)$ , because

$$\begin{aligned} k(t; k_0, c_0) &= k(t; k_0, 0) > 0 \quad \forall t \in [0, \bar{t}] \\ k(t + \bar{t}; k_0, c_0) &= k(t; k_1, c_1) \geq 0 \quad \forall t \geq 0 \end{aligned}$$

by the uniqueness of the orbit; as far as the second equality is concerned, observe that both orbits pass through  $(0, k_1)$  and satisfy the differential equation controlled with  $c_1$  for  $t > 0$ . Hence by (34):

$$\begin{aligned} V(k_1) - V(k_0) &\leq U(c_1; k_1) - U(c_0; k_0) = \int_0^{+\infty} e^{-\rho t} [u(c_1(t)) - u(c_0(t))] dt \\ &= \int_0^{+\infty} e^{-\rho t} u(c_1(t)) dt - \int_{\bar{t}}^{+\infty} e^{-\rho t} u(c_1(t - \bar{t})) dt \\ &= \int_0^{+\infty} e^{-\rho t} u(c_1(t)) dt - \int_0^{+\infty} e^{-\rho(s+\bar{t})} u(c_1(s)) ds \\ &= (1 - e^{-\rho \bar{t}}) U(c_1; k_1) = (1 - e^{-\rho \bar{t}}) V(k_1) \leq \rho \bar{t} V(k_1) < \rho V(k_1) \frac{k_1 - k_0}{F(k_0)} \end{aligned}$$

So by the monotonicity of  $V$  and  $F$  we have, for  $a \leq k_0 < k_1 \leq b$ :

$$V(k_1) - V(k_0) \leq \rho \frac{V(b)}{F(a)} (k_1 - k_0).$$

□

## 7 Dynamic Programming

In this section we study the properties of the value function as a solution to Bellman and Hamilton-Jacobi-Bellman equations.

First observe that we can translate an orbit by translating the control, according to the next remark.

*Remark 26* (Translation of the orbit). For every  $k_0 \geq 0$  and every  $c \in \mathcal{L}_{loc}^1((0, +\infty), \mathbb{R})$ :

$$k(\cdot; k(\tau; k_0, c), c(\cdot + \tau)) = k(\cdot + \tau; k_0, c)$$

by the uniqueness of the orbit. In particular, if  $c \in \Lambda(k_0)$  then  $c(\cdot + \tau) \in \Lambda(k(\tau; k_0, c))$ .

The first step consists in proving a suitable version of Dynamic Programming Principle.

**Theorem 27 (Bellman's Dynamic Programming Principle).** *For every  $\tau > 0$ , the value function  $V : [0, +\infty) \rightarrow \mathbb{R}$  satisfies the following functional equation:*

$$\forall k_0 \geq 0 : v(k_0) = \sup_{c \in \Lambda(k_0)} \left\{ \int_0^\tau e^{-\rho t} u(c(t)) dt + e^{-\rho \tau} v(k(\tau; k_0, c)) \right\} \quad (35)$$

in the unknown  $v : [0, +\infty) \rightarrow \mathbb{R}$ .

*Proof.* Fix  $\tau > 0$  and  $k_0 \geq 0$ , and set

$$\sigma(\tau, k_0) := \sup_{c \in \Lambda(k_0)} \left\{ \int_0^\tau e^{-\rho t} u(c(t)) dt + e^{-\rho \tau} V(k(\tau; k_0, c)) \right\}.$$

We prove that

$$\sigma(\tau, k_0) = \sup_{c \in \Lambda(k_0)} U(c; k_0).$$

In the first place, we show that  $\sigma(\tau, k_0)$  is an upper bound of  $\{U(c; k_0) / c \in \Lambda(k_0)\}$ .

Fix  $c \in \Lambda(k_0)$ ; then by Remark 26  $c(\cdot + \tau) \in \Lambda(k(\tau; k_0, c))$ ; hence

$$\begin{aligned} \sigma(\tau, k_0) &\geq \int_0^\tau e^{-\rho t} u(c(t)) dt + e^{-\rho \tau} V(k(\tau; k_0, c)) \\ &\geq \int_0^\tau e^{-\rho t} u(c(t)) dt + e^{-\rho \tau} U(c(\cdot + \tau); k(\tau; k_0, c)) \\ &= \int_0^\tau e^{-\rho t} u(c(t)) dt + \int_0^{+\infty} e^{-\rho(t+\tau)} u(c(t+\tau)) dt \\ &= \int_0^\tau e^{-\rho t} u(c(t)) dt + \int_\tau^{+\infty} e^{-\rho s} u(c(s)) ds = U(c; k_0) \end{aligned}$$

In the second place, fix  $\epsilon > 0$ , and take

$$0 < \epsilon' \leq \frac{2\epsilon}{(1 + e^{-\rho \tau})}.$$

Hence there exists  $\tilde{c}_\epsilon \in \Lambda(k_0)$  and  $\tilde{\tilde{c}}_\epsilon \in \Lambda(k(\tau; k_0, \tilde{c}_\epsilon))$  such that

$$\begin{aligned} \sigma(\tau, k_0) - \epsilon &\leq \sigma(\tau, k_0) - \frac{\epsilon'}{2} (1 + e^{-\rho\tau}) \\ &\leq \int_0^\tau e^{-\rho t} u(\tilde{c}_\epsilon(t)) dt + e^{-\rho\tau} V(k(\tau; k_0, \tilde{c}_\epsilon)) - e^{-\rho\tau} \frac{\epsilon'}{2} \\ &\leq \int_0^\tau e^{-\rho t} u(\tilde{c}_\epsilon(t)) dt + e^{-\rho\tau} U(\tilde{\tilde{c}}_\epsilon; k(\tau; k_0, \tilde{c}_\epsilon)) \\ &= \int_0^\tau e^{-\rho t} u(\tilde{c}_\epsilon(t)) dt + \int_0^{+\infty} e^{-\rho(t+\tau)} u(\tilde{\tilde{c}}_\epsilon(t)) dt \end{aligned}$$

Now set

$$c_\epsilon(t) := \begin{cases} \tilde{c}_\epsilon(t) & \text{if } t \in [0, \tau] \\ \tilde{\tilde{c}}_\epsilon(t - \tau) & \text{if } t > \tau \end{cases}$$

Hence  $c_\epsilon \in \mathcal{L}_{loc}^1((0, +\infty), \mathbb{R})$  and  $\forall t > 0 : c_\epsilon(t + \tau) = \tilde{\tilde{c}}_\epsilon(t)$ . So:

$$\sigma(\tau, k_0) - \epsilon \leq \int_0^{+\infty} e^{-\rho t} u(c_\epsilon(t)) dt \quad (36)$$

Finally, it is easy to show that  $c_\epsilon \in \Lambda(k_0)$ . Observe that  $k(\cdot; k_0, c_\epsilon) = k(\cdot; k_0, \tilde{c}_\epsilon)$  in  $[0, \tau]$  by definition of  $c_\epsilon$  and by uniqueness. In particular  $k(\tau; k_0, c_\epsilon) = k(\tau; k_0, \tilde{c}_\epsilon)$ , so that  $k(\cdot + \tau; k_0, c_\epsilon)$  and  $k(\cdot; k(\tau; k_0, \tilde{c}_\epsilon), \tilde{\tilde{c}}_\epsilon)$  have the same initial value; moreover, these two orbits satisfy the same state equation (i.e. the equation associated to the control  $c_\epsilon(\cdot + \tau)$ ) and so they coincide, again by uniqueness. Recalling that by definition  $\tilde{c}_\epsilon \in \Lambda(k_0)$  and  $\tilde{\tilde{c}}_\epsilon \in \Lambda(k(\tau; k_0, \tilde{c}_\epsilon))$ , we have  $k(t; k_0, c_\epsilon) \geq 0$  for all  $t \geq 0$ . Hence by (36) we can write

$$\sigma(\tau, k_0) - \epsilon \leq U(c_\epsilon; k_0)$$

and the assertion is proven.  $\square$

Equation (35) is called *Bellman Functional Equation*.

A consequence of the above theorem is that every control which is optimal respect to a state, is also optimal respect to every following optimal state.

**Corollary 28.** *Let  $k_0 \geq 0$ ,  $c^* \in \Lambda(k_0)$ . Hence the following are equivalent:*

- i)  $c^*$  is optimal at  $k_0$
- ii) For every  $\tau > 0$ :

$$V(k_0) = \int_0^\tau e^{-\rho t} u(c^*(t)) dt + e^{-\rho\tau} V(k(\tau; k_0, c^*))$$

Moreover, i) or ii) imply that for every  $\tau > 0$ ,  $c^*(\cdot + \tau)$  is admissible and optimal at  $k(\tau; k_0, c^*)$ .

*Proof.* i)  $\Rightarrow$  ii) Let us assume that  $c^*$  is admissible and optimal at  $k_0 \geq 0$  and fix  $\tau > 0$ . Observe

that  $c^*(\cdot + \tau)$  is admissible at  $k(\tau; k_0, c^*)$  by Remark 26. Hence, by Theorem 27:

$$\begin{aligned} V(k_0) &\geq \int_0^\tau e^{-\rho t} u(c^*(t)) dt + e^{-\rho \tau} V(k(\tau; k_0, c^*)) \\ &\geq \int_0^\tau e^{-\rho t} u(c^*(t)) dt + e^{-\rho \tau} U(c^*(\cdot + \tau); k(\tau; k_0, c^*)) \\ &= \int_0^{+\infty} e^{-\rho t} u(c^*(t)) dt = U(c^*; k_0) = V(k_0). \end{aligned} \quad (37)$$

Hence

$$V(k_0) = \int_0^\tau e^{-\rho t} u(c^*(t)) dt + e^{-\rho \tau} V(k(\tau; k_0, c^*)). \quad (38)$$

ii)  $\Rightarrow$  i) Suppose that  $c^* \in \Lambda(k_0)$  and (38) holds for every  $\tau > 0$ . For every  $\epsilon > 0$  pick  $\hat{c}_\epsilon \in \Lambda(k(\frac{1}{\epsilon}; k_0, c^*))$  such that:

$$V\left(k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right) - \epsilon \leq U\left(\hat{c}_\epsilon; k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right). \quad (39)$$

Then define

$$c_\epsilon(t) := \begin{cases} c^*(t) & \text{if } t \in [0, \frac{1}{\epsilon}] \\ \hat{c}_\epsilon(t - \frac{1}{\epsilon}) & \text{if } t > \frac{1}{\epsilon} \end{cases}$$

By the same arguments we used in the proof of Theorem 27,  $c_\epsilon \in \Lambda(k_0)$  and, obviously,  $c_\epsilon(t + \frac{1}{\epsilon}) = \hat{c}_\epsilon(t)$  for every  $t > 0$ .

Hence, taking  $\tau = 1/\epsilon$  in (38), we have by (39):

$$\begin{aligned} V(k_0) - \epsilon e^{-\rho/\epsilon} &= \int_0^{1/\epsilon} e^{-\rho t} u(c^*(t)) dt + e^{-\rho/\epsilon} \left[ V\left(k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right) - \epsilon \right] \\ &\leq \int_0^{1/\epsilon} e^{-\rho t} u(c^*(t)) dt + e^{-\rho/\epsilon} U\left(\hat{c}_\epsilon; k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right) \\ &= \int_0^{1/\epsilon} e^{-\rho t} u(c^*(t)) dt + \int_0^{+\infty} e^{-\rho(t+\frac{1}{\epsilon})} u\left(c_\epsilon\left(t + \frac{1}{\epsilon}\right)\right) dt \\ &= \int_0^{1/\epsilon} e^{-\rho t} u(c^*(t)) dt + \int_{1/\epsilon}^{+\infty} e^{-\rho s} u(c_\epsilon(s)) ds \end{aligned} \quad (40)$$

Now we show that the second addend tends to 0 as  $\epsilon \rightarrow 0$ . Observe that by Jensen inequality, for every  $T \geq 1/\epsilon$ :

$$\begin{aligned} \int_{1/\epsilon}^T e^{-\rho s} u(c_\epsilon(s)) ds &= \left[ e^{-\rho s} \int_{1/\epsilon}^s u(c_\epsilon(\tau)) d\tau \right]_{s=1/\epsilon}^{s=T} + \rho \int_{1/\epsilon}^T e^{-\rho s} \int_{1/\epsilon}^s u(c_\epsilon(\tau)) d\tau ds \\ &\leq e^{-\rho T} \int_0^T u(c_\epsilon(\tau)) d\tau + \rho \int_{1/\epsilon}^T e^{-\rho s} \int_0^s u(c_\epsilon(\tau)) d\tau ds \\ &\leq e^{-\rho T} \int_0^T u(c_\epsilon(\tau)) d\tau + \rho \int_{1/\epsilon}^T s e^{-\rho s} u\left(\frac{\int_0^s c_\epsilon(\tau) d\tau}{s}\right) ds \\ &\rightarrow \rho \int_{1/\epsilon}^{+\infty} s e^{-\rho s} u\left(\frac{\int_0^s c_\epsilon(\tau) d\tau}{s}\right) ds \quad \text{as } T \rightarrow +\infty, \end{aligned} \quad (41)$$

by Lemma 16, ii) and by the admissibility of  $c_\epsilon$ . By point i) of the same Lemma, for every  $\epsilon < 1$  and every  $s \geq 1/\epsilon$ :

$$\begin{aligned} se^{-\rho s} u \left( \frac{\int_0^s c_\epsilon(\tau) d\tau}{s} \right) &\leq se^{-\rho s} u \left( M(k_0) \left[ 1 + e^{(L+\epsilon_0)s} \right] + \frac{M(k_0)}{s(L+\epsilon_0)} \right) \\ &\leq se^{-\rho s} \left\{ u(M(k_0)) + M(k_0) u \left( e^{(L+\epsilon_0)s} \right) + u \left( \frac{M(k_0)}{L+\epsilon_0} \right) \right\} \end{aligned}$$

which implies, together with (41), for every  $\epsilon < 1$ :

$$\begin{aligned} 0 \leq \int_{1/\epsilon}^{+\infty} e^{-\rho s} u(c_\epsilon(s)) ds &\leq \rho \int_{1/\epsilon}^{+\infty} se^{-\rho s} u \left( \frac{\int_0^s c_\epsilon(\tau) d\tau}{s} \right) ds \\ &\leq \rho \left[ u(M(k_0)) + u \left( \frac{M(k_0)}{L+\epsilon_0} \right) \right] \int_{1/\epsilon}^{+\infty} se^{-\rho s} ds + \\ &\quad + \rho M(k_0) \int_{1/\epsilon}^{+\infty} se^{-\rho s} u \left( e^{(L+\epsilon_0)s} \right) ds. \end{aligned}$$

By Remark 3 this quantity tends to 0 as  $\epsilon \rightarrow 0$ .

Hence, letting  $\epsilon \rightarrow 0$  in (40), we find:

$$V(k_0) \leq \int_0^{+\infty} e^{-\rho t} u(c^*(t)) dt = U(c^*; k_0)$$

which implies that  $c^*$  is optimal at  $k_0$ .

Finally, if i) holds, then by (37):

$$V(k(\tau; k_0, c^*)) = U(c^*(\cdot + \tau); k(\tau; k_0, c^*)).$$

□

A careful study of the difference quotients for the functions

$$t \rightarrow e^{-\rho t} V(k(t))$$

(for an orbit  $k$ ) leads to the following definitions and theorems.

**Definition 29.** Let  $f \in \mathcal{C}^0((0, +\infty), \mathbb{R})$ ; we say that  $f \in \mathcal{C}^+((0, +\infty), \mathbb{R})$  if, and only if, for every  $k_0 > 0$  there exist  $\delta, C^+, C^- > 0$  such that

$$\begin{aligned} \frac{f(k_0 + h) - f(k_0)}{h} &\geq C^+ \quad \forall h \in (0, \delta) \\ \frac{f(k_0 + h) - f(k_0)}{h} &\geq C^- \quad \forall h \in (-\delta, 0) \end{aligned}$$

We note that by Theorem 25, (ii) the value function  $V$  satisfies

$$V \in \mathcal{C}^+((0, +\infty), \mathbb{R}). \quad (42)$$

**Definition 30.** The function  $H : [0, +\infty) \times (0, +\infty) \rightarrow \mathbb{R}$  defined by

$$H(k, p) := -\sup \{ [F(k) - c] \cdot p + u(c) \mid c \in [0, +\infty) \}$$

is called *Hamiltonian*.

The equation

$$\rho v(k) + H(k, v'(k)) = 0 \quad \forall k > 0 \quad (43)$$

in the unknown  $v \in \mathcal{C}^+((0, +\infty), \mathbb{R}) \cap \mathcal{C}^1((0, +\infty), \mathbb{R})$  is called *Hamilton-Jacobi-Bellman equation* (HJB).

Observe that any solution of (43) must be strictly increasing, by Definition 29.

*Remark 31.* The Hamiltonian is always finite. Indeed

$$-\sup_{c \in [0, +\infty)} \{ [F(k) - c] \cdot p + u(c) \} > -\infty \iff p > 0.$$

If  $p > 0$ , since  $\lim_{c \rightarrow +\infty} u'(c) = 0$  we can choose  $c_p \geq 0$  such that  $u'(c_p) \leq p$ ; this implies by the concavity of  $u$ :

$$\forall c \geq 0 : u(c) - cp \leq u(c) - u'(c_p)c \leq u(c_p) - u'(c_p)c_p,$$

so that

$$-F(k)p - \sup_{c \in [0, +\infty)} \{ u(c) - cp \} \geq -F(k)p - u(c_p) + u'(c_p)c_p > -\infty.$$

Otherwise, when  $p \leq 0$ , since  $\lim_{c \rightarrow +\infty} u(c) = +\infty$  we have

$$-F(k)p - \sup_{c \in [0, +\infty)} \{ u(c) - cp \} \leq -F(k)p - \sup_{c \in [0, +\infty)} u(c) = -\infty.$$

**Definition 32.** A function  $v \in \mathcal{C}^+((0, +\infty), \mathbb{R})$  is called a *viscosity subsolution* [*supersolution*] of (HJB) if, and only if:

for every  $\varphi \in \mathcal{C}^1((0, +\infty), \mathbb{R})$  and for every local maximum [minimum] point  $k_0 > 0$  of  $v - \varphi$ :

$$\begin{aligned} \rho v(k_0) - \sup \{ [F(k_0) - c] \cdot \varphi'(k_0) + u(c) \mid c \in [0, +\infty) \} &= \\ \rho v(k_0) + H(k_0, \varphi'(k_0)) &\leq 0 \\ &[\geq 0] \end{aligned}$$

If  $v$  is both a viscosity subsolution of (HJB) and a viscosity supersolution of (HJB), then we say that  $v$  is a *viscosity solution* of (HJB).

*Remark 33.* The latter definition is well posed. Indeed, let  $v \in \mathcal{C}^+((0, +\infty), \mathbb{R})$  and  $\varphi \in \mathcal{C}^1((0, +\infty), \mathbb{R})$ . If  $k_0$  is a local maximum for  $v - \varphi$  in  $(0, +\infty)$ , then for  $h < 0$  big enough we have:

$$\begin{aligned} v(k_0) - v(k_0 + h) &\geq \varphi(k_0) - \varphi(k_0 + h) \implies \\ 0 < C^- &\leq \frac{v(k_0) - v(k_0 + h)}{h} \leq \frac{\varphi(k_0) - \varphi(k_0 + h)}{h}. \end{aligned}$$

If  $k_0$  is a local minimum for  $v - \varphi$  in  $(0, +\infty)$ , then for  $h > 0$  small enough we have:

$$\begin{aligned} v(k_0) - v(k_0 + h) &\leq \varphi(k_0) - \varphi(k_0 + h) \implies \\ 0 < C^+ &\leq \frac{v(k_0) - v(k_0 + h)}{h} \leq \frac{\varphi(k_0) - \varphi(k_0 + h)}{h}. \end{aligned}$$

In both cases, we have  $\varphi'(k_0) > 0$ .

We are now going to prove that the value function is a viscosity solution of (HJB). As pointed out in the introduction, this will be done without any regularity assumption on  $H$ ; nevertheless, this function can be easily shown to be continuous, since for every  $k \geq 0$ ,  $p > 0$ :

$$H(k, p) = F(k)p + (-u)^*(p),$$

where  $(-u)^*$  is the (convex) conjugate function of the convex function  $-u$ .

**Lemma 34.** *Let  $k_0 > 0$  and  $(c_T)_{T>0} \subseteq \Lambda(k_0)$  satisfying:*

$$\|c_T\|_{\infty, [0, T]} \leq N(k_0, T) \quad \forall T > 0.$$

where  $N$  is the function defined in Lemma 9. Hence

$$\forall T \in [0, 1] : \forall t \in [0, T] : |k(t; k_0, c_T) - k_0| \leq Te^{\bar{M}t} [F(k_0) + N(k_0, 1)].$$

In particular  $k(T; k_0, c_T) \rightarrow k_0$  as  $T \rightarrow 0$ .

*Proof.* Set  $k_0$  and  $(c_T)_{T>0}$  as in the hypothesis and fix  $0 \leq T \leq 1$ . Hence integrating both sides of the state equation we get, for every  $t \in [0, T]$ :

$$k(t; k_0, c_T) - k_0 = \int_0^t [F(k_0) - c_T(s)] ds + \int_0^t [F(k(s; k_0, c_T)) - F(k_0)] ds$$

which implies by Remark 7:

$$\begin{aligned} |k(t; k_0, c_T) - k_0| &\leq \int_0^t |F(k_0) - c_T(s)| ds + \int_0^t |F(k(s; k_0, c_T)) - F(k_0)| ds \\ &\leq \int_0^T |F(k_0) - c_T(s)| ds + \bar{M} \int_0^t |k(s; k_0, c_T) - k_0| ds \end{aligned}$$

Hence by Gronwall's inequality and by the monotonicity of  $N(k_0, \cdot)$ , for every  $T \in [0, 1]$  and every  $t \in [0, T]$ :

$$\begin{aligned} |k(t; k_0, c_T) - k_0| &\leq e^{\bar{M}t} \int_0^T |F(k_0) - c_T(s)| ds. \\ &\leq Te^{\bar{M}t} [F(k_0) + N(k_0, T)] \\ &\leq Te^{\bar{M}t} [F(k_0) + N(k_0, 1)]. \end{aligned}$$

□

**Proposition 35.** *The value function  $V : [0, +\infty) \rightarrow \mathbb{R}$  is a viscosity solution of (HJB).*

*Consequently, if  $V \in \mathcal{C}^1([0, +\infty), \mathbb{R})$ , then  $V$  is strictly increasing and is a solution of (HJB) - (43) in the classical sense.*

*Proof.* In the first place, we show that  $V$  is a viscosity supersolution of (HJB).

Let  $\varphi \in \mathcal{C}^1((0, +\infty), \mathbb{R})$  and  $k_0 > 0$  be a local minimum point of  $V - \varphi$ , so that

$$V(k_0) - V \leq \varphi(k_0) - \varphi \quad (44)$$

in a proper neighbourhood of  $k_0$ . Now fix  $c \in [0, +\infty)$  and set  $k := k(\cdot; k_0, c)$ . As  $k_0 > 0$ , there exists  $T_c > 0$  such that  $k > 0$  in  $[0, T_c]$ . Hence the control

$$\tilde{c}(t) := \begin{cases} c & \text{if } t \in [0, T_c] \\ 0 & \text{if } t > T_c \end{cases}$$

is admissible at  $k_0$ . Then by Theorem 27, for every  $\tau \in [0, T_c]$ :

$$\begin{aligned} V(k_0) - V(k(\tau)) &\geq \int_0^\tau e^{-\rho t} u(\tilde{c}(t)) dt + V(k(\tau)) [e^{-\rho\tau} - 1] \\ &= u(c) \int_0^\tau e^{-\rho t} dt + V(k(\tau)) [e^{-\rho\tau} - 1]. \end{aligned}$$

Hence by (44) and by the continuity of  $k$ , we have for every  $\tau > 0$  sufficiently small:

$$\frac{\varphi(k(0)) - \varphi(k(\tau))}{\tau} \geq u(c) \frac{\int_0^\tau e^{-\rho t} dt}{\tau} + V(k(\tau)) \frac{[e^{-\rho\tau} - 1]}{\tau}.$$

Letting  $\tau \rightarrow 0$  and using the continuity of  $V$  and  $k$ :

$$-\varphi'(k_0) [F(k_0) - c] \geq u(c) - \rho V(k_0)$$

which implies, taking the sup for  $c \geq 0$ :

$$\rho V(k_0) + H(k_0, \varphi'(k_0)) \geq 0$$

Secondly we show that  $V$  is a viscosity subsolution of (HJB).

Let  $\varphi \in \mathcal{C}^1((0, +\infty), \mathbb{R})$  and  $k_0 > 0$  be a local maximum point of  $V - \varphi$ , so that

$$V(k_0) - V \geq \varphi(k_0) - \varphi \quad (45)$$

in a proper neighborhood  $\mathcal{N}(k_0)$  of  $k_0$ .

Fix  $\epsilon > 0$  and, using the definition of  $V$ , define a family of controls  $(c_{T,\epsilon})_{T>0} \subseteq \Lambda(k_0)$  such that for every  $T > 0$ :

$$V(k_0) - T\epsilon \leq U(c_{T,\epsilon}; k_0). \quad (46)$$



Now take  $(c_{T,\epsilon})^T$  as in Lemma 9 and set  $\bar{c}_{T,\epsilon} := (c_{T,\epsilon})^T$  for simplicity of notation (so that  $\bar{c}_{T,\epsilon} \in \Lambda(k_0)$ ). We have:

$$\begin{aligned} V(k_0) - T\epsilon &\leq U(c_{T,\epsilon}; k_0) \leq U(\bar{c}_{T,\epsilon}; k_0) \\ &= \int_0^T e^{-\rho t} u(\bar{c}_{T,\epsilon}(t)) dt + e^{-\rho T} \int_T^{+\infty} e^{-\rho(s-T)} u(\bar{c}_{T,\epsilon}(s-T+T)) ds \\ &= \int_0^T e^{-\rho t} u(\bar{c}_{T,\epsilon}(t)) dt + e^{-\rho T} U(\bar{c}_{T,\epsilon}(\cdot+T); k(T; k_0, \bar{c}_{T,\epsilon})) \\ &\leq \int_0^T e^{-\rho t} u(\bar{c}_{T,\epsilon}(t)) dt + e^{-\rho T} V(k(T; k_0, \bar{c}_{T,\epsilon})) \end{aligned}$$

where we have used Remark 26.

By Lemma 34 we have for  $T > 0$  sufficiently small (say  $T < \hat{T}$ ),

$$k(T; k_0, \bar{c}_{T,\epsilon}) \in \mathcal{N}(k_0).$$

Hence, setting  $\bar{k}_{T,\epsilon} := k(\cdot; k_0, \bar{c}_{T,\epsilon})$ , for every  $T < \hat{T}$ , we have by (45):

$$\begin{aligned} \varphi(k_0) - \varphi(\bar{k}_{T,\epsilon}(T)) - e^{-\rho T} V(\bar{k}_{T,\epsilon}(T)) &\leq V(k_0) - V(\bar{k}_{T,\epsilon}(T)) - e^{-\rho T} V(\bar{k}_{T,\epsilon}(T)) \\ &\leq \int_0^T e^{-\rho t} u(\bar{c}_{T,\epsilon}(t)) dt - V(\bar{k}_{T,\epsilon}(T)) + T\epsilon \end{aligned}$$

which implies

$$\begin{aligned} &\int_0^T -\{\varphi'(\bar{k}_{T,\epsilon}(t)) [F(\bar{k}_{T,\epsilon}(t)) - \bar{c}_{T,\epsilon}(t)] + e^{-\rho t} u(\bar{c}_{T,\epsilon}(t))\} dt \\ &\leq V(\bar{k}_{T,\epsilon}(T)) [e^{-\rho T} - 1] + T\epsilon. \end{aligned} \tag{47}$$

Observe that the integral at the left hand member bigger than:

$$\begin{aligned} &\int_0^T -\{\varphi'(k_0) + \omega_1(t) [F(k_0) - \bar{c}_{T,\epsilon}(t) + \omega_2(t)] + u(\bar{c}_{T,\epsilon}(t))\} dt = \\ &\int_0^T -\{\varphi'(k_0) [F(k_0) - \bar{c}_{T,\epsilon}(t)] + u(\bar{c}_{T,\epsilon}(t))\} dt + \\ &+ \int_0^T -\{\varphi'(k_0) \omega_2(t) + \omega_1(t) [\omega_2(t) + F(k_0) - \bar{c}_{T,\epsilon}(t)]\} dt \end{aligned} \tag{48}$$

where  $\omega_1, \omega_2$  are functions which are continuous in a neighborhood of 0 and satisfy:

$$\omega_1(0) = \omega_2(0) = 0.$$

This implies, for  $T < 1$ :

$$\begin{aligned} &\left| \int_0^T \varphi'(k_0) \omega_2(t) dt + \int_0^T \omega_1(t) [\omega_2(t) + F(k_0) - \bar{c}_{T,\epsilon}(t)] dt \right| \\ &\leq |\varphi'(k_0)| o_1(T) + o_2(T) + \int_0^T |\omega_1(t)| [F(k_0) + \bar{c}_{T,\epsilon}(t)] dt \\ &\leq |\varphi'(k_0)| o_1(T) + o_2(T) + [F(k_0) + N(k_0, T)] o_3(T) \\ &\leq |\varphi'(k_0)| o_1(T) + o_2(T) + [F(k_0) + N(k_0, 1)] o_3(T) \end{aligned}$$

where

$$\lim_{T \rightarrow 0} \frac{o_i(T)}{T} = 0$$

for  $i = 1, 2, 3$ . Observe that this is true even if the  $o_i$ s depend on  $T$ , by Lemma 34. For instance,

$$\begin{aligned} |o_1(T)| &= \left| \int_0^T \omega_2(t) dt \right| \leq T \max_{[0,T]} |\omega_2| = T |\omega_2(\tau_T)| \\ &= T |F(\bar{k}_{T,\epsilon}(\tau_T)) - F(k_0)| \\ &\leq \bar{M}T |\bar{k}_{T,\epsilon}(\tau_T) - k_0| \leq \bar{M}T^2 e^{\bar{M}\tau_T} [F(k_0) + N(k_0, 1)] \end{aligned}$$

Moreover, by the fact that  $V \in C^+([0, +\infty), \mathbb{R})$  and by Remark 33, we have for any  $t \in [0, T]$ :

$$\begin{aligned} -\{\varphi'(k_0)[F(k_0) - \bar{c}_{T,\epsilon}(t)] + u(\bar{c}_{T,\epsilon}(t))\} &\geq -\sup_{c \geq 0} \{\varphi'(k_0)[F(k_0) - c] + u(c)\} \\ &= H(k_0, \varphi'(k_0)) > -\infty, \end{aligned}$$

by which we can write:

$$\int_0^T -\{\varphi'(k_0)[F(k_0) - \bar{c}_{T,\epsilon}(t)] + u(\bar{c}_{T,\epsilon}(t))\} dt \geq T \cdot H(k_0, \varphi'(k_0)).$$

Hence, by (47) and (48):

$$\begin{aligned} &V(\bar{k}_{T,\epsilon}(T)) [e^{-\rho T} - 1] + T\epsilon \\ &\geq -\int_0^T \{\varphi'(k_0)[F(k_0) - \bar{c}_{T,\epsilon}(t)] + u(\bar{c}_{T,\epsilon}(t))\} dt + \\ &\quad + \int_0^T -\{\varphi'(k_0)\omega_2(t) dt + \omega_1(t)[\omega_2(t) + F(k_0) - \bar{c}_{T,\epsilon}(t)] dt\} \\ &\geq T \cdot H(k_0, \varphi'(k_0)) + o_{T \rightarrow 0}(T) \end{aligned}$$

for any  $0 < T < 1, \hat{T}$ . Hence dividing by  $T$ , and then letting  $T \rightarrow 0$ , again by Lemma 34 and the continuity of  $V$  we obtain:

$$-\rho V(k_0) + \epsilon \geq H(k_0, \varphi'(k_0))$$

which proves the assertion since  $\epsilon$  is arbitrary.  $\square$

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